

Theoretical and empirical results for recovery from multiple measurements*

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Abstract

The joint-sparse recovery problem aims to recover, from sets of compressed measurements, unknown sparse matrices with nonzero entries restricted to a subset of rows. This is an extension of the single-measurement-vector (SMV) problem widely studied in compressed sensing. We study the recovery properties of two algorithms for problems with noiseless data and exact-sparse representation. First, we show that recovery using sum-of-norm minimization cannot exceed the uniform-recovery rate of sequential SMV using ℓ_1 minimization, and that there are problems that can be solved with one approach but not the other. Second, we study the performance of the ReMBo algorithm [M. Mishali and Y. Eldar, *IEEE Trans. Sig. Proc.*, 56 (2008)] in combination with ℓ_1 minimization, and show how recovery improves as more measurements are taken. From this analysis it follows that having more measurements than the number of linearly independent nonzero rows does not improve the potential theoretical recovery rate.

1 Introduction

A problem of central importance in compressed sensing [1,9] is the following: given an $m \times n$ matrix A , and a measurement vector $b = Ax_0$, recover x_0 . When $m < n$, this problem is ill-posed, and it is not generally possible to uniquely recover x_0 without some prior information. In many important cases, x_0 is known to be sparse, and it may be appropriate to solve

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad \|x\|_0 \quad \text{subject to} \quad Ax = b, \quad (1.1)$$

to find the sparsest possible solution. (The ℓ_0 -norm $\|\cdot\|_0$ of a vector counts the number of nonzero entries.) If x_0 has fewer than $s/2$ nonzero entries, where s is the number of nonzeros in the sparsest null-vector of A , then x_0 is the unique solution of this optimization problem [11, 19]. The main obstacle of this approach is that it is combinatorial [24], and therefore impractical for all but the smallest problems. To overcome this, Chen et al. [5] introduced basis pursuit:

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad \|x\|_1 \quad \text{subject to} \quad Ax = b. \quad (1.2)$$

This convex relaxation, based on the ℓ_1 -norm $\|x\|_1$, can be solved much more efficiently; moreover, under certain conditions [2, 10], it yields the same solution as the ℓ_0 problem (1.1).

A natural extension of the single-measurement-vector (SMV) problem just described is the multiple-measurement-vector (MMV) problem. Instead of a single measurement b , we are given a set of r measurements

$$b^{(k)} = Ax_0^{(k)}, \quad k = 1, \dots, r,$$

in which the vectors $x_0^{(k)}$ are jointly sparse—i.e., have nonzero entries concentrated at common locations. Such problems arise in source localization [22], neuromagnetic imaging [7], and equalization of sparse-communication channels [6, 15]. Succinctly, the aim of the MMV problem is to

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recover X_0 from observations $B = AX_0$, where $B = [b^{(1)}, b^{(2)}, \dots, b^{(r)}]$ is an $m \times r$ matrix, and the $n \times r$ matrix X_0 is *row sparse*—i.e., it has nonzero entries in only a small number of rows.

The most widely studied approach to the MMV problem is based on solving the convex optimization problem

$$\underset{X \in \mathbb{R}^{n \times r}}{\text{minimize}} \quad \|X\|_{p,q} \quad \text{subject to} \quad AX = B,$$

where the mixed $\ell_{p,q}$ norm of X is defined as

$$\|X\|_{p,q} = \left(\sum_{j=1}^n \|X^{j\rightarrow}\|_q^p \right)^{1/p},$$

and $X^{j\rightarrow}$ is the (column) vector whose entries form the j th row of X . In particular, Cotter et al. [7] consider $p = 2, q \leq 1$; Tropp [28, 29] analyzes $p = 1, q = \infty$; Malioutov et al. [22] and Eldar and Mishali [13] use $p = 1, q = 2$; and Chen and Huo [4] study $p = 1, q \geq 1$. A different approach is given by Mishali and Eldar [23], who propose the ReMBo algorithm, which reduces MMV to a series of SMV problems. We denote the combination of ReMBo and an ℓ_1 solver for the SMV subproblem as ReMBo- ℓ_1 .

In this paper we study the sum-of-norms problem and the conditions for uniform recovery of all X_0 with a fixed row support, and compare this against recovery using $\ell_{1,1}$. We then construct matrices X_0 that cannot be recovered using $\ell_{1,1}$ but for which $\ell_{1,2}$ does succeed, and vice versa. We then illustrate the individual recovery properties of $\ell_{1,1}$ and $\ell_{1,2}$ with empirical results. We further show how recovery via $\ell_{1,1}$ can only degrade as the number of measurements increases, and show how boosting [23] can improve on the $\ell_{1,1}$ approach. This analysis, which applies only to noiseless data with exact-sparse representation, provides the starting point for our study of the recovery properties of ReMBo- ℓ_1 , and is based on a geometrical interpretation of this algorithm.

We begin in Section 2 by summarizing existing ℓ_0 - ℓ_1 equivalence results, which give conditions under which the solution of the ℓ_1 relaxation (1.2) coincides with the solution of the ℓ_0 problem (1.1). In Section 3 we consider the $\ell_{1,2}$ mixed-norm and sum-of-norms formulations and compare their performance against $\ell_{1,1}$. In Sections 4 and 5 we examine two approaches that are based on sequential application of (1.2).

Assumptions. We make the following assumptions throughout. The matrix $A \in \mathbb{R}^{m \times n}$ is full-rank. The unknown matrix to be recovered, $X_0 \in \mathbb{R}^{n \times r}$, is s row-sparse. Except for our study of ReMBo, the columns of X_0 have identical support and exactly s sparse.

Notation. We follow the convention that all vectors are column vectors. For an arbitrary matrix M , its j th column is denoted by the column vector $M^{j\leftarrow}$; its i th row is the transpose of the column vector $M^{i\rightarrow}$. The i th entry of a vector v is denoted by v_i . We make exceptions for $e_i = I^{i\leftarrow}$ and for x_0 (resp., X_0), which represents the sparse vector (resp., matrix) we want to recover. When there is no ambiguity we sometimes write m_i to denote $M^{i\leftarrow}$. When concatenating vectors into matrices, $[a, b, c]$ denotes horizontal concatenation and $[a; b; c]$ denotes vertical concatenation. When indexing with \mathcal{I} , we define the vector $v_{\mathcal{I}} := [v_i]_{i \in \mathcal{I}}$, and the $m \times |\mathcal{I}|$ matrix $A_{\mathcal{I}} := [A^{i\leftarrow}]_{i \in \mathcal{I}}$. Row or column selection takes precedence over all other operators.

2 Existing results for ℓ_1 recovery

The conditions under which (1.2) gives the sparsest possible solution have been studied by applying a number of different techniques. By far the most popular analytical approach is based on the restricted isometry property, introduced by Candès and Tao [3], which gives sufficient conditions for equivalence. Donoho [8] obtains necessary and sufficient (NS) conditions by analyzing the underlying geometry of (1.2). Several authors [11, 12, 19] characterize the NS-conditions in terms of properties of the kernel of A :

$$\text{Ker}(A) = \{x \mid Ax = 0\}.$$

Fuchs [16, Theorem 4] and Tropp [27] express sufficient conditions in terms of the solution of the dual of (1.2):

$$\underset{y}{\text{maximize}} \quad b^T y \quad \text{subject to} \quad \|A^T y\|_\infty \leq 1. \quad (2.1)$$

In this paper we are mainly concerned with the geometric and kernel conditions. We use the geometrical interpretation of the problems to get a better understanding, and resort to the null-space properties of A to analyze recovery. To make the discussion more self-contained, we briefly recall some of the relevant results in the next three sections.

2.1 The geometry of ℓ_1 recovery

The set of all points of the unit ℓ_1 -ball, $\{x \in \mathbb{R}^n \mid \|x\|_1 \leq 1\}$, can be formed by taking convex combinations of $\pm e_j$, the signed columns of the identity matrix. Geometrically this is equivalent to taking the convex hull of these vectors, giving the cross-polytope $\mathcal{C} = \text{conv}\{\pm e_1, \pm e_2, \dots, \pm e_n\}$. Likewise, we can look at the linear mapping $x \mapsto Ax$ for all points $x \in \mathcal{C}$, giving the polytope $\mathcal{P} = \{Ax \mid x \in \mathcal{C}\} = A\mathcal{C}$. The faces of \mathcal{C} can be expressed as the convex hull of subsets of vertices, not including pairs that are reflections with respect to the origin (such pairs are sometimes erroneously referred to as antipodal, which is a slightly more general concept [21]). Under linear transformations, each face from the cross-polytope \mathcal{C} either maps to a face on \mathcal{P} or vanishes into the interior of \mathcal{P} .

The solution found by (1.2) can be interpreted as follows. Starting with a radius of zero, we slowly “inflate” \mathcal{P} until it first touches b . The radius at which this happens corresponds to the ℓ_1 -norm of the solution x^* . The vertices whose convex hull is the face touching b determine the location and sign of the non-zero entries of x^* , while the position where b touches the face determines their relative weights. Donoho [8] shows that x_0 can be recovered from $b = Ax_0$ using (1.2) if and only if the face of the (scaled) cross-polytope containing x_0 maps to a face on \mathcal{P} . Two direct consequences are that recovery depends only on the sign pattern of x_0 , and that the probability of recovering a random s -sparse vector is equal to the ratio of the number of $(s-1)$ -faces in \mathcal{P} to the number of $(s-1)$ -faces in \mathcal{C} . That is, letting $\mathcal{F}_d(\mathcal{P})$ denote the collection of all d -faces [21] in \mathcal{P} , the probability of recovering x_0 using ℓ_1 is given by

$$P_{\ell_1}(A, s) = \frac{|\mathcal{F}_{s-1}(A\mathcal{C})|}{|\mathcal{F}_{s-1}(\mathcal{C})|}.$$

When we need to find the recoverability of vectors with support \mathcal{I} , this probability becomes

$$P_{\ell_1}(A, \mathcal{I}) = \frac{|\mathcal{F}_{\mathcal{I}}(A\mathcal{C})|}{|\mathcal{F}_{\mathcal{I}}(\mathcal{C})|}, \quad (2.2)$$

where $\mathcal{F}_{\mathcal{I}}(\mathcal{C}) = 2^{|\mathcal{I}|}$ denotes the number of faces in \mathcal{C} formed by the convex hull of $\{\pm e_j\}_{j \in \mathcal{I}}$, and $\mathcal{F}_{\mathcal{I}}(A\mathcal{C})$ is the number of faces on $A\mathcal{C}$ generated by $\{\pm A^{ij}\}_{j \in \mathcal{I}}$.

2.2 Null-space properties and ℓ_1 recovery

Equivalence results in terms of null-space properties generally characterize equivalence for the set of all vectors x with a fixed support, which is defined as

$$\text{Supp}(x) = \{j \mid x_j \neq 0\}.$$

We say that x can be uniformly recovered on $\mathcal{I} \subseteq \{1, \dots, n\}$ if all x with $\text{Supp}(x) \subseteq \mathcal{I}$ can be recovered. The following theorem illustrates conditions for uniform recovery via ℓ_1 on an index set; more general results are given by Gribonval and Nielsen [20].

Theorem 2.1 (Donoho and Elad [11], Gribonval and Nielsen [19]). *Let A be an $m \times n$ matrix and $\mathcal{I} \subseteq \{1, \dots, n\}$ be a fixed index set. Then all $x_0 \in \mathbb{R}^n$ with $\text{Supp}(x_0) \subseteq \mathcal{I}$ can be uniquely recovered from $b = Ax_0$ using basis pursuit (1.2) if and only if for all $z \in \text{Ker}(A) \setminus \{0\}$,*

$$\sum_{j \in \mathcal{I}} |z_j| < \sum_{j \notin \mathcal{I}} |z_j|. \quad (2.3)$$

That is, the ℓ_1 -norm of z on \mathcal{I} is strictly less than the ℓ_1 -norm of z on the complement \mathcal{I}^c .

2.3 Optimality conditions for ℓ_1 recovery

Sufficient conditions for recovery can be derived from the first-order optimality conditions necessary for x^* and y^* to be solutions of (1.2) and (2.1) respectively. The Karush-Kuhn-Tucker (KKT) conditions are also sufficient in this case because the problems are convex. The Lagrangian function for (1.2) is given by

$$\mathcal{L}(x, y) = \|x\|_1 - y^T(Ax - b);$$

the KKT conditions require that

$$Ax = b \quad \text{and} \quad 0 \in \partial_x \mathcal{L}(x, y), \quad (2.4)$$

where $\partial_x \mathcal{L}$ denotes the subdifferential of \mathcal{L} with respect to x . The second condition reduces to

$$0 \in \text{sgn}(x) - A^T y,$$

where the signum function

$$\text{sgn}(\gamma) \in \begin{cases} \text{sign}(\gamma) & \text{if } \gamma \neq 0, \\ [-1, 1] & \text{otherwise,} \end{cases}$$

is applied to each individual component of x . It follows that x^* is a solution of (1.2) if and only if $Ax^* = b$ and there exists an m -vector y such that $|a_j^T y| \leq 1$ for $j \notin \text{Supp}(x)$, and $a_j^T y = \text{sign}(x_j^*)$ for all $j \in \text{Supp}(x)$. Fuchs [16] shows that x^* is the unique solution of (1.2) when $[a_j]_{j \in \text{Supp}(x)}$ is full rank and, in addition, $|a_j^T y| < 1$ for all $j \notin \text{Supp}(x)$. When the columns of A are in general position (i.e., no $i + 1$ columns of A span the same $i - 1$ dimensional hyperplane for $i \leq n$) we can weaken this condition by noting that for such A , the solution of (1.2) is always unique, thus making the existence of a y that satisfies (2.4) for x_0 a necessary and sufficient condition for ℓ_1 to recover x_0 .

3 Recovery using sums-of-row norms

Our analysis of sparse recovery for the MMV problem of recovering X_0 from $B = AX_0$ begins with an extension of Theorem 2.1 to recovery using the convex relaxation

$$\underset{X}{\text{minimize}} \quad \sum_{j=1}^n \|X^{j\rightarrow}\| \quad \text{subject to} \quad AX = B; \quad (3.1)$$

note that the norm within the summation is arbitrary. Define the row support of a matrix as

$$\text{Supp}_{\text{row}}(X) = \{j \mid \|X^{j\rightarrow}\| \neq 0\}.$$

With these definitions we have the following result. (A related result is given by Stojnic et al. [26].)

Theorem 3.1. *Let A be an $m \times n$ matrix, $\mathcal{I} \subseteq \{1, \dots, n\}$ be a fixed index set, and let $\|\cdot\|$ denote any vector norm. Then all $X_0 \in \mathbb{R}^{n \times r}$ with $\text{Supp}_{\text{row}}(X_0) \subseteq \mathcal{I}$ can be uniquely recovered from $B = AX_0$ using (3.1) if and only if for all $Z \neq 0$ with columns $Z^{i_k} \in \text{Ker}(A)$,*

$$\sum_{j \in \mathcal{I}} \|Z^{j\rightarrow}\| < \sum_{j \notin \mathcal{I}} \|Z^{j\rightarrow}\|. \quad (3.2)$$

Proof. For the ‘‘only if’’ part, suppose that there is a $Z \neq 0$ with columns $Z^{i_k} \in \text{Ker}(A)$ such that (3.2) does not hold. Now, choose $X^{j\rightarrow} = Z^{j\rightarrow}$ for all $j \in \mathcal{I}$ and with all remaining rows zero. Set $B = AX$. Next, define $V = X - Z$, and note that $AV = AX - AZ = AX = B$. The construction of V implies that $\sum_j \|X^{j\rightarrow}\| \geq \sum_j \|V^{j\rightarrow}\|$, and consequently X cannot be the unique solution of (3.1).

Conversely, let X be an arbitrary matrix with $\text{Supp}_{\text{row}}(X) \subseteq \mathcal{I}$, and let $B = AX$. To show that X is the unique solution of (3.1) it suffices to show that for any Z with columns $Z^{1k} \in \text{Ker}(A) \setminus \{0\}$,

$$\sum_j \|(X + Z)^{j\rightarrow}\| > \sum_j \|X^{j\rightarrow}\|.$$

This is equivalent to

$$\sum_{j \notin \mathcal{I}} \|Z^{j\rightarrow}\| + \sum_{j \in \mathcal{I}} \|(X + Z)^{j\rightarrow}\| - \sum_{j \in \mathcal{I}} \|X^{j\rightarrow}\| > 0.$$

Applying the reverse triangle inequality, $\|a + b\| - \|b\| \geq -\|a\|$, to the summation over $j \in \mathcal{I}$ and reordering exactly gives condition (3.2). \square

Earlier results by Chen and Huo [4, Theorem 3.1] give sufficient conditions, based on strict feasibility of a dual problem, for unique recovery.

In the special case of the sum of ℓ_1 -norms, i.e., $\ell_{1,1}$, summing the norms of the columns is equivalent to summing the norms of the rows. As a result, (3.1) can be written as

$$\underset{X}{\text{minimize}} \quad \sum_{k=1}^r \|X^{1k}\|_1 \quad \text{subject to} \quad AX^{1k} = B^{1k}, \quad k = 1, \dots, r. \quad (3.3)$$

Because this objective is separable (as observed by Tropp [28]) the problem can be decoupled and solved as a series of independent basis pursuit problems, giving one X^{1k} for each column B^{1k} of B . The following result relates recovery using the sum-of-norms formulation (3.1) to $\ell_{1,1}$ recovery.

Theorem 3.2. *Let A be an $m \times n$ matrix, $\mathcal{I} \subseteq \{1, \dots, n\}$ be a fixed index set, and $\|\cdot\|$ denote any vector norm. Then uniform recovery of all $X \in \mathbb{R}^{n \times r}$ with $\text{Supp}_{\text{row}}(X) \subseteq \mathcal{I}$ using sums of norms (3.1) implies uniform recovery on \mathcal{I} using $\ell_{1,1}$.*

Proof. For uniform recovery on support \mathcal{I} to hold it follows from Theorem 3.1 that for any matrix $Z \neq 0$ with columns $Z^{1k} \in \text{Ker}(A)$, property (3.2) holds. In particular it holds for Z with $Z^{1k} = \bar{z}$ for all k , with $\bar{z} \in \text{Ker}(A) \setminus \{0\}$. Note that for these matrices there exist a norm-dependent constant γ such that

$$|\bar{z}_j| = \gamma \|Z^{j\rightarrow}\|.$$

Since the choice of \bar{z} was arbitrary, it follows from (3.2) that the NS-condition (2.3) for independent recovery of vectors B^{1k} using ℓ_1 in Theorem 2.1 is satisfied. Moreover, because $\ell_{1,1}$ is equivalent to independent recovery, we also have uniform recovery on \mathcal{I} using $\ell_{1,1}$. \square

An implication of Theorem 3.2 is that the use of restricted isometry conditions—or any technique, for that matter—to analyze uniform recovery conditions for the sum-of-norms approach necessarily lead to results that are no stronger than uniform ℓ_1 recovery. (Recall that the $\ell_{1,1}$ and ℓ_1 norms are equivalent). Eldar and Rauhut [14, Prop. 4.1] make a similar observation with regard to recovery using the $\ell_{1,2}$ norm. Their result can easily be extended to the general sum-of-norms formulation.

3.1 Recovery using $\ell_{1,2}$

In this section we take a closer look at the $\ell_{1,2}$ problem

$$\underset{X}{\text{minimize}} \quad \|X\|_{1,2} \quad \text{subject to} \quad AX = B, \quad (3.4)$$

which is a special case of the sum-of-norms problem. Although Theorem 3.2 establishes that uniform recovery via $\ell_{1,2}$ is no better than uniform recovery via $\ell_{1,1}$, there are many situations in which it recovers signals that $\ell_{1,1}$ cannot. Indeed, it is evident from Figure 1 that the probability of recovering individual signals with random signs and support is much higher for $\ell_{1,2}$. This confirms the theoretical results by Eldar and Rauhut [14], who show that in the average case, the recovery rate of $\ell_{1,2}$ benefits from multiple observations, i.e., increasing r . In contrast, the performance of $\ell_{1,1}$ in Figure 1 is seen to degrade with increasing r . We explain the reason behind this in Section 4.

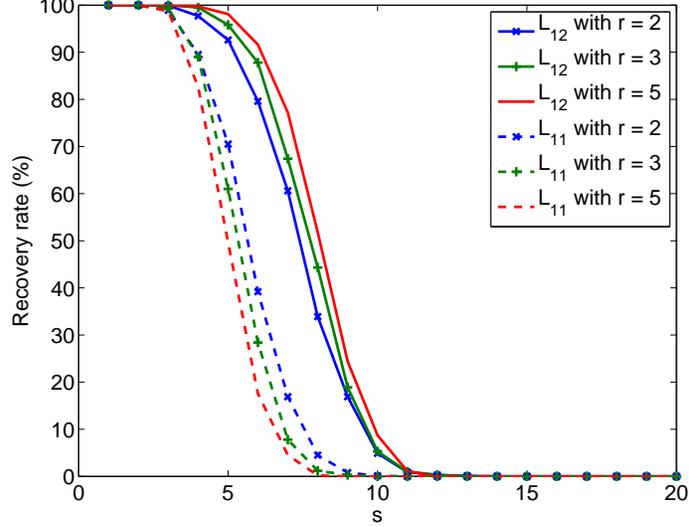


Figure 1: Recovery rates for fixed, randomly drawn 20×60 matrices A , averaged over 1,000 trials at each row-sparsity level s . The nonzero entries in the $60 \times r$ matrix X_0 are sampled i.i.d. from the normal distribution. The solid and dashed lines represent $\ell_{1,2}$ and $\ell_{1,1}$ recovery, respectively.

In this section we construct examples for which $\ell_{1,2}$ works and $\ell_{1,1}$ fails, and vice versa. This helps uncover some of the structure of $\ell_{1,2}$, but at the same time implies that certain techniques used to study ℓ_1 can no longer be used directly. Because the examples are based on extensions of the results from Section 2.3, we first develop equivalent conditions here.

3.1.1 Sufficient conditions for recovery via $\ell_{1,2}$

The optimality conditions of the $\ell_{1,2}$ problem (3.4) play a vital role in deriving a set of sufficient conditions for joint-sparse recovery. In this section we derive the dual of (3.4) and the corresponding necessary and sufficient optimality conditions. These allow us to derive sufficient conditions for recovery via $\ell_{1,2}$.

The Lagrangian for (3.4) is defined as

$$\mathcal{L}(X, Y) = \|X\|_{1,2} - \langle Y, AX - B \rangle, \quad (3.5)$$

where $\langle V, W \rangle := \text{trace}(V^T W)$ is an inner-product defined over real matrices. The dual is then given by maximizing

$$\inf_X \mathcal{L}(X, Y) = \inf_X \{ \|X\|_{1,2} - \langle Y, AX - B \rangle \} = \langle B, Y \rangle - \sup_X \{ \langle A^T Y, X \rangle - \|X\|_{1,2} \} \quad (3.6)$$

over Y . (Because the primal problem has only linear constraints, there necessarily exists a dual solution Y^* that maximizes this expression [25, Theorem 28.2].) To simplify the supremum term, we note that for any convex, positively homogeneous function f defined over an inner-product space,

$$\sup_v \{ \langle w, v \rangle - f(v) \} = \begin{cases} 0 & \text{if } w \in \partial f(0), \\ \infty & \text{otherwise.} \end{cases}$$

To derive these conditions, note that positive homogeneity of f implies that $f(0) = 0$, and thus $w \in \partial f(0)$ implies that $\langle w, v \rangle \leq f(v)$ for all v . Hence, the supremum is achieved with $v = 0$. If on the other hand $w \notin \partial f(0)$, then there exists some v such that $\langle w, v \rangle > f(v)$, and by the positive homogeneity of f , $\langle w, \alpha v \rangle - f(\alpha v) \rightarrow \infty$ as $\alpha \rightarrow \infty$. Applying this expression for the supremum to (3.6), we arrive at the necessary condition

$$A^T Y \in \partial \|0\|_{1,2}, \quad (3.7)$$

which is required for dual feasibility.

We now derive an expression for the subdifferential $\partial\|X\|_{1,2}$. For rows j where $\|X^{j\rightarrow}\|_2 > 0$, the gradient is given by $\nabla\|X^{j\rightarrow}\|_2 = X^{j\rightarrow}/\|X^{j\rightarrow}\|_2$. For the remaining rows, the gradient is not defined, but $\partial\|X^{j\rightarrow}\|_2$ coincides with the set of unit ℓ_2 -norm vectors $\mathcal{B}_{\ell_2}^r = \{v \in \mathbb{R}^r \mid \|v\|_2 \leq 1\}$. Thus, for each $j = 1, \dots, n$,

$$\partial_{X^{j\rightarrow}}\|X\|_{1,2} \in \begin{cases} X^{j\rightarrow}/\|X^{j\rightarrow}\|_2 & \text{if } \|X^{j\rightarrow}\|_2 > 0, \\ \mathcal{B}_{\ell_2}^r & \text{otherwise.} \end{cases} \quad (3.8)$$

Combining this expression with (3.7), we arrive at the dual of (3.4):

$$\underset{Y}{\text{maximize}} \quad \text{trace}(B^T Y) \quad \text{subject to} \quad \|A^T Y\|_{\infty,2} \leq 1. \quad (3.9)$$

The following conditions are therefore necessary and sufficient for a primal-dual pair (X^*, Y^*) to be optimal for (3.4) and its dual (3.9):

$$AX^* = B \quad (\text{primal feasibility}); \quad (3.10a)$$

$$\|A^T Y^*\|_{\infty,2} \leq 1 \quad (\text{dual feasibility}); \quad (3.10b)$$

$$\|X^*\|_{1,2} = \text{trace}(B^T Y^*) \quad (\text{zero duality gap}). \quad (3.10c)$$

The existence of a matrix Y^* that satisfies (3.10) provides a certificate that the feasible matrix X^* is an optimal solution of (3.4). However, it does not guarantee that X^* is also the unique solution. The following theorem gives sufficient conditions, similar to those in Section 2.3, that also guarantee uniqueness of the solution.

Theorem 3.3. *Let A be an $m \times n$ matrix, and B be an $m \times r$ matrix. Then a set of sufficient conditions for X to be the unique minimizer of (3.4) with Lagrange multiplier $Y \in \mathbb{R}^{m \times r}$ and row support $\mathcal{I} = \text{Supp}_{\text{row}}(X)$, is that*

$$AX = B, \quad (3.11a)$$

$$(A^T Y)^{j\rightarrow} = (X^*)^{j\rightarrow}/\|(X^*)^{j\rightarrow}\|_2, \quad j \in \mathcal{I} \quad (3.11b)$$

$$\|(A^T Y)^{j\rightarrow}\|_2 < 1, \quad j \notin \mathcal{I} \quad (3.11c)$$

$$\text{rank}(A_{\mathcal{I}}) = |\mathcal{I}|. \quad (3.11d)$$

Proof. The first three conditions clearly imply that (X, Y) primal and dual feasible, and thus satisfy (3.10a) and (3.10b). Conditions (3.11b) and (3.11c) together imply that

$$\text{trace}(B^T Y) \equiv \sum_{j=1}^n [(A^T Y)^{j\rightarrow}]^T X^{j\rightarrow} = \sum_{j=1}^n \|X^{j\rightarrow}\|_2 \equiv \|X\|_{1,2}.$$

The first and last identities above follow directly from the definitions of the matrix trace and of the norm $\|\cdot\|_{1,2}$, respectively; the middle equality follows from the standard Cauchy inequality. Thus, the zero-gap requirement (3.10c) is satisfied. The conditions (3.11a)–(3.11c) are therefore sufficient for (X, Y) to be an optimal primal-dual solution of (3.4). Because Y determines the support and is a Lagrange multiplier for every solution X , this support must be unique. It then follows from condition (3.11d) that X must be unique. \square

3.2 Counter examples

Using the sufficient and necessary conditions developed in the previous section we now construct examples of problems for which $\ell_{1,2}$ succeeds while $\ell_{1,1}$ fails, and vice versa. Because of its simplicity, we begin with the latter.

3.2.1 Recovery using $\ell_{1,1}$ where $\ell_{1,2}$ fails.

Consider the matrices

$$A = \begin{bmatrix} 1 & 0.5 & 1 \\ 0 & 0.5 & 0.8 \end{bmatrix}, \quad \text{and} \quad X_0 = \begin{bmatrix} 2 & 1 \\ 2 & 10 \\ 0 & 0 \end{bmatrix}.$$

By drawing AC , the convex hull of the columns of $\pm A$, it is easily seen that convex combinations of the first two columns give points on a face of the polytope. Because the weights in the columns of X_0 are scalar multiples of such points they can be uniquely recovered using ℓ_1 minimization, and consequently X_0 itself can be recovered using $\ell_{1,1}$.

On the other hand, for $\ell_{1,2}$ minimization to recover X_0 , there must exist a $Y \in \mathbb{R}^{2 \times 2}$ satisfying both (3.10b) and (3.11b). However, the unique Y satisfying the latter condition does not satisfy the former, thereby showing that $\ell_{1,2}$ fails to recover X_0 .

3.2.2 Recovery using $\ell_{1,2}$ where $\ell_{1,1}$ fails.

For the construction of a problem where $\ell_{1,2}$ succeeds and $\ell_{1,1}$ fails, we consider two vectors, f and s , with the same support \mathcal{I} , in such a way that individual ℓ_1 recovery fails for f , while it succeeds for s . In addition we assume that there exists a vector y that satisfies

$$y^T A^{:j} = \text{sign}(s_j) \quad \text{for all } j \in \mathcal{I}, \quad \text{and} \quad |y^T A^{:j}| < 1 \quad \text{for all } j \notin \mathcal{I};$$

i.e., y satisfies conditions (3.11b) and (3.11c). Using the vectors f and s , we construct the 2-column matrix $X_0 = [(1 - \gamma)s, \gamma f]$, and claim that for sufficiently small $\gamma > 0$, this gives the desired reconstruction problem. Clearly, for any $\gamma \neq 0$, $\ell_{1,1}$ recovery fails because the second column can never be recovered, and we only need to show that $\ell_{1,2}$ does succeed.

For $\gamma = 0$, the matrix $Y = [y, 0]$ satisfies conditions (3.11b) and (3.11c) and, assuming (3.11d) is also satisfied, X_0 is the unique solution of $\ell_{1,2}$ with $B = AX_0$. For sufficiently small $\gamma > 0$, the conditions that Y need to satisfy change slightly due to the division by $\|X_0^{j^-}\|_2$ for those rows in $\text{Supp}_{\text{row}}(X)$. By adding corrections to the columns of Y those new conditions can be satisfied. In particular, these corrections can be done by adding weighted combinations of the columns in \bar{Y} , which are constructed in such a way that it satisfies $A_{\mathcal{I}^c}^T \bar{Y} = I$, and minimizes $\|A_{\mathcal{I}^c}^T \bar{Y}\|_{\infty, \infty}$ on the complement \mathcal{I}^c of \mathcal{I} .

Note that on the above argument can also be used to show that $\ell_{1,2}$ fails for γ sufficiently close to one. Because the support and signs of X remain the same for all $0 < \gamma < 1$, we can conclude the following: *recovery using $\ell_{1,2}$ can be influenced by the magnitude of the nonzero entries of X_0 . This is unlike $\ell_{1,1}$, where recovery depends only on the support and sign pattern of the nonzero entries.* A consequence of this conclusion is that the notion of faces used in the geometrical interpretation of ℓ_1 is not applicable to the $\ell_{1,2}$ problem.

3.3 Experiments

To get an idea of just how much more $\ell_{1,2}$ can recover in the above case where $\ell_{1,1}$ fails, we generated a 20×60 matrix A with entries i.i.d. normally distributed, and determined a set of vectors s_i and f_i with identical support for which ℓ_1 recovery succeeds and fails, respectively. Using triples of vectors s_i and f_j we constructed row-sparse matrices such as $X_0 = [s_1, f_1, f_2]$ or $X_0 = [s_1, s_2, f_2]$, and attempted to recover from $B = AX_0 W$, where $W = \text{diag}(\omega_1, \omega_2, \omega_3)$ is a diagonal weighting matrix with nonnegative entries and unit trace, by solving (3.4). For problems of this size, interior-point methods are very efficient and we use SDPT3 [30] through the CVX interface [17, 18]. We consider X_0 to be recovered when the maximum absolute difference between X_0 and the $\ell_{1,2}$ solution X^* is less than 10^{-5} . The results of the experiment are shown in Figure 2. In addition to the expected regions of recovery around individual columns s_i and failure around f_i , we see that certain combinations of vectors s_i still fail, while other combinations of vectors f_i may be recoverable. By contrast, when using $\ell_{1,1}$ to solve the problem, any combination of s_i vectors can be recovered while no combination including an f_i can be recovered.

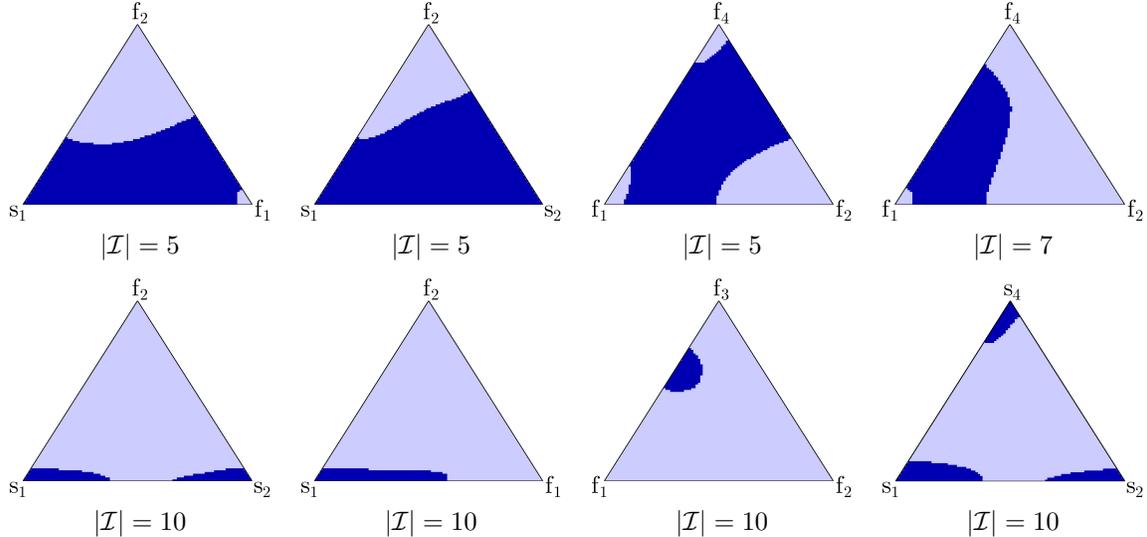


Figure 2: Generation of problems where $\ell_{1,2}$ succeeds, while $\ell_{1,1}$ fails. For a 20×60 matrix A and fixed support of size $|\mathcal{I}| = 5, 7, 10$, we create vectors f_i that cannot be recovered using ℓ_1 , and vectors s_i than can be recovered. Each triangle represents an X_0 constructed from the vectors denoted in the corners. The location in the triangle determines the weight on each vector, ranging from zero to one, and summing up to one. The dark areas indicates the weights for which $\ell_{1,2}$ successfully recovered X_0 .

4 Bridging the gap from $\ell_{1,1}$ to ReMBo

We begin this section with a discussion showing that the performance of $\ell_{1,1}$ can only get worse with increasing number of observations, thus explaining empirical observations made earlier by Chen and Huo [4] and Mishali and Eldar [23]. We then show how the recovery rate can be improved by using the boosting technique introduced by Mishali and Eldar [23]. The resulting boosted- ℓ_1 approach is a simplified version of the ReMBo- ℓ_1 algorithm, which we discuss in section 5. Although boosted- ℓ_1 has a lower performance, we include it because its simplicity makes it easy to analyze and allows us to show more intuitively what it is that makes ReMBo- ℓ_1 work so well. Also, the recovery rate of boosted- ℓ_1 motives a performance model for ReMBo- ℓ_1 recovery. Thus, boosted- ℓ_1 —although not a viable algorithm in practice—bridges the gap between $\ell_{1,1}$ and ReMBo- ℓ_1 .

As described in Section 3, recovery using $\ell_{1,1}$ —i.e., (3.3)—is equivalent to individual ℓ_1 recovery of each column $x^{(k)} := X_0^{1:k}$ based on solving (1.2) with $b := B^{1:k}$, for $k = 1, \dots, r$. Assume that the signs of nonzero entries in the support of each $x^{(k)}$ are uniformly distributed. Then we can express the probability of recovering X_0 with row support \mathcal{I} using $\ell_{1,1}$ in terms of the probability of recovering individual vectors on that support using ℓ_1 . By the separability of (3.3), $\ell_{1,1}$ recovers X_0 if and only if (1.2) successfully recovers each $x^{(k)}$. Denote the recovery rate of an $x^{(k)}$ supported on \mathcal{I} by $P_{\ell_1}(A, \mathcal{I})$. Then the expected $\ell_{1,1}$ recovery rate is

$$P_{\ell_{1,1}}(A, \mathcal{I}, r) = [P_{\ell_1}(A, \mathcal{I})]^r.$$

This expression shows that the probability of recovery using $\ell_{1,1}$ can only decrease as r increases, which clearly defeats the purpose of gathering multiple observations; see Figure 1.

There are many problem instances where $\ell_{1,1}$ fails to recover X_0 as a whole but does correctly recover a subset of columns $x^{(k)}$. The following boosting procedure [23] exploits this fact and uses it to help generate the entire solution. Given such a vector $x^{(k)}$ with support \mathcal{J} of sufficiently small cardinality (e.g., less than $m/2$), solve the following system for \bar{X} :

$$\underset{\bar{X}}{\text{minimize}} \quad \|A_{\mathcal{J}}\bar{X} - B\|_F. \quad (4.1)$$

```

given  $A, B$ 
for  $k = 1, \dots, r$  do
  solve (1.2) with  $b := B^{lk}$  to get  $x$ 
   $\mathcal{J} \leftarrow \text{Supp}(x)$ 
  if  $|\mathcal{J}| < m/2$  then
    solve (4.1) to get  $\bar{X}$ 
    if  $A_{\mathcal{J}}\bar{X} = B$  then
       $X^* = 0$ 
       $[(X^*)^{j-}]_{j \in \mathcal{J}} \leftarrow \bar{X}$ 
      return solution  $X^*$ 
  return failure

```

Figure 3: The boosted ℓ_1 algorithm

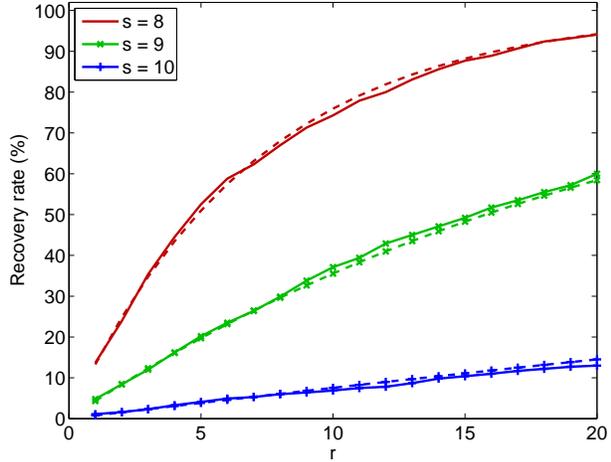


Figure 4: Theoretical (dashed) and experimental (solid) performance of boosted ℓ_1 for three problem instances with different row support s .

If the residual in (4.1) is zero, conclude that the support \mathcal{J} coincides with \mathcal{I} and assume that the nonzero entries of X_0 are given by \bar{X} . If the residual is nonzero, the support \mathcal{J} is necessarily incorrect, and the next sufficiently-sparse vector is checked. This approach is outlined in Figure 3.

The recovery properties of the boosted ℓ_1 approach are opposite to those of $\ell_{1,1}$: it fails only if all individual columns with support \mathcal{I} fail to be recovered using ℓ_1 . Hence, given an unknown $n \times r$ matrix X_0 supported on \mathcal{I} with its sign pattern uniformly random, the boosted ℓ_1 algorithm enjoys an expected recovery rate of

$$P(A, \mathcal{I}, r) = 1 - [1 - P_{\ell_1}(A, \mathcal{I})]^r. \quad (4.2)$$

This result hinges on our blanket assumption that the columns of X_0 have identical support.

To experimentally verify this recovery rate, we generate a 20×80 matrix A with entries independently sampled from the normal distribution and fix a randomly chosen support set \mathcal{I}_s for three levels of sparsity, $s = 8, 9, 10$. On each of these three supports we generate vectors with all possible sign patterns and solve (1.2) to verify if they can be recovered (see Section 3.3). This gives exactly the face counts required to compute the ℓ_1 recovery probability in (2.2), and the expected boosted ℓ_1 recovery rate in (4.2).

For the empirical success rate we take the average over 1,000 trials with random matrices X_0 supported on \mathcal{I}_s , and its nonzero entries independently drawn from the normal distribution. Because recovery of individual vectors using ℓ_1 minimization depends only on their sign pattern, we reduce the computational time by comparing the sign patterns against precomputed recovery tables (this is possible because both A and \mathcal{I}_s remain fixed), rather than invoking an ℓ_1 solver for each vector. The theoretical and empirical recovery rates using boosted ℓ_1 are plotted in Figure 4.

5 Recovery using ReMBo

The ReMBo algorithm by Mishali and Eldar [23] proceeds by taking a random r -vector w and combining the individual observations in B into a single weighted observation $b := Bw$. It then solves an SMV problem $Ax = b$ and checks if the computed solution x^* is sufficiently sparse. If not, the above steps are repeated with a different weight vector w ; the algorithm stops when a maximum number of trials is reached. If the support \mathcal{J} of x^* is small, form $A_{\mathcal{J}}$ and check if (4.1) has a solution \bar{X} with zero residual. In this case we have the nonzero rows of the solution X^* in \bar{X} and are done. Otherwise, we simply proceed with the next w .

```

given  $A, B$ . Set  $\text{Iteration} \leftarrow 0$ 
while  $\text{Iteration} < \text{MaxIteration}$  do
   $w \leftarrow \text{Random}(n, 1)$ 
  solve (1.2) with  $b = Bw$  to get  $x$ 
   $\mathcal{J} \leftarrow \text{Supp}(x)$ 
  if  $|\mathcal{J}| < m/2$  then
    solve (4.1) to get  $\bar{X}$ 
    if  $A_{\mathcal{J}}\bar{X} = B$  then
       $\bar{X}^* = 0$ 
       $[(X^*)^{j \rightarrow}]_{j \in \mathcal{J}} \leftarrow \bar{X}$ 
      return solution  $X^*$ 
     $\text{Iteration} \leftarrow \text{Iteration} + 1$ 
return failure

```

Figure 5: The ReMBo- ℓ_1 algorithm

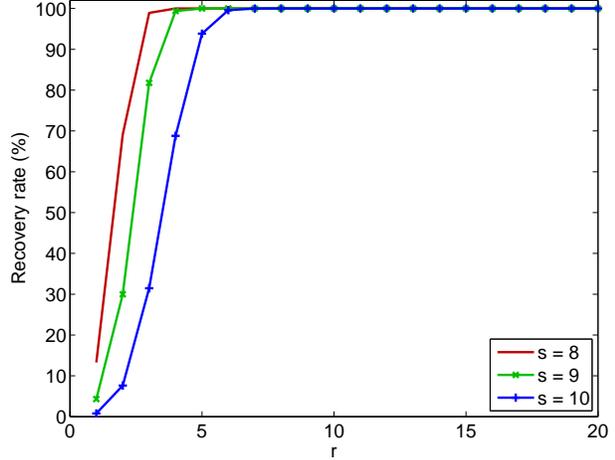


Figure 6: Theoretical performance model for ReMBo on three problem instances with different sparsity levels s .

The ReMBo algorithm does not prescribe a particular SMV solver. However, throughout this paper we use ReMBo in conjunction with ℓ_1 minimization, thus giving the ReMBo- ℓ_1 algorithm summarized in Figure 5. The formulation given in [23] requires a user-defined threshold on the cardinality of the support \mathcal{J} instead of the fixed threshold $m/2$. Ideally this threshold should be half of the spark of A , where

$$\text{Spark}(A) := \min_{z \in \text{Ker}(A) \setminus \{0\}} \|z\|_0, \quad (5.1)$$

is the number of nonzeros of the sparsest vector in the kernel of A ; any vector x_0 with fewer than $\text{Spark}(A)/2$ nonzeros is the unique sparsest solution of $Ax = Ax_0 = b$ [11]. Unfortunately, the spark is prohibitively expensive to compute, but under the assumption that A is in general position, $\text{Spark}(A) = m + 1$. Note that choosing a higher value can help to recover signals with row sparsity exceeding $m/2$. However, in this case it can no longer be guaranteed to be the sparsest solution.

In our study of ReMBo- ℓ_1 , we fix an unknown matrix X_0 with row support \mathcal{I} of cardinality s . We deviate from our blanket assumption (cf. Section 1) and allow for individual columns to be supported on a subset of \mathcal{I} . Each time we multiply B by a random weight vector $w^{(k)}$, we in fact create a new problem which, with probability one, has an exact s -sparse solution $x_0 := X_0 w^{(k)}$.

As reflected in (2.2), recovery of x_0 using ℓ_1 depends only on its support and sign pattern. Clearly, the probability of recovery improves as the number of distinct sign patterns encountered by ReMBo- ℓ_1 increases. The maximum number of sign patterns encountered with boosted ℓ_1 is the number of observations r . The question thus becomes, how many different sign patterns ReMBo- ℓ_1 can encounter by taking linear combinations of the columns in X_0 ? (We disregard the situation where elimination occurs and $|\text{Supp}(X_0 w)| < s$.) Equivalently, we can ask how many orthants in \mathbb{R}^s (each corresponding to a different sign pattern) can be properly intersected by the subspace given by the range of the submatrix \bar{X} consisting of the nonzero rows of X_0 (with proper we mean intersection of the interior). In Section 5.1 we derive an exact expression for the maximum number of proper orthant intersections in \mathbb{R}^n by a subspace generated by d vectors, denoted by $C(n, d)$.

Based on the above reasoning, a good model for a bound on the recovery rate for $n \times r$ matrices X_0 with $\text{Supp}_{\text{row}}(X_0) = \mathcal{I} < m/2$ using ReMBo- ℓ_1 is given by

$$P_R(A, \mathcal{I}, r) = 1 - \prod_{i=1}^S \left[1 - \frac{\mathcal{F}_{\mathcal{I}}(AC)}{\mathcal{F}_{\mathcal{I}}(C) - 2(i-1)} \right], \quad (5.2)$$

where S denotes the number of unique sign patterns tried. The maximum possible value of S is $C(|\mathcal{I}|, r)/2$; for subspaces spanned by the columns of X_0 , with normally distributed nonzero

entries, intersects $C(|\mathcal{I}|, r)$ unique orthants with probability one (cf. Corollary 5.3). The term within brackets denotes the probability of failure and the fraction represents the success rate, which is given by the ratio of the number of faces $\mathcal{F}_{\mathcal{I}}(\mathcal{AC})$ that survived the mapping to the total number of faces to consider. The total number reduces by two at each trial because we can exclude the face f we just tried, as well as $-f$. The factor of two in $C(|\mathcal{I}|, r)/2$ is also due to this symmetry¹.

This model would be a bound for the average performance of ReMBo- ℓ_1 if the sign patterns generated would be randomly sampled from the space of all sign patterns on the given support. However, because they are generated from the orthant intersections with a subspace, the actual set of patterns is highly structured. Indeed, it is possible to imagine a situation where the $(s-1)$ -faces in \mathcal{C} that perish in the mapping to \mathcal{AC} have sign patterns that are all contained in the set generated by a single subspace. Any other set of sign patterns would then necessarily include some faces that survive the mapping and by trying all patterns in that set we would recover X_0 . In this case, the average recovery over all X_0 on that support could be much higher than that given by (5.2). An interesting question is how the surviving faces of \mathcal{C} are distributed. Due to the simplicial structure of the facets of \mathcal{C} , we can expect the faces that perish to be partially clustered (if a $(d-2)$ -face perishes, then so will the two $(d-1)$ -faces whose intersection gives this face), and partially unclustered (the faces that perish while all their sub-faces survive). Note that, regardless of these patterns, recovery is guaranteed in the limit whenever the number of unique sign patterns tried exceeds half the number of faces lost, $(|\mathcal{F}_{\mathcal{I}}(\mathcal{C})| - |\mathcal{F}_{\mathcal{I}}(\mathcal{AC})|)/2$.

Figure 6 illustrates the theoretical performance model based on $C(n, d)$, for which we derive the exact expression in Section 5.1. In Section 5.2 we discuss practical limitations, and in Section 5.3 we empirically look at how the number of sign patterns generated grows with the number of normally distributed vectors w , and how this affects the recovery rates. To allow comparison between ReMBo and boosted ℓ_1 , we used the same matrix A and support \mathcal{I}_s used to generate Figure 4.

5.1 Maximum number of orthant intersections with a subspace

Theorem 5.1. *Let $C(n, d)$ denote the maximum attainable number of orthant interiors intersected by a subspace in \mathbb{R}^n generated by d vectors. Then $C(n, 1) = 2$, $C(n, d) = 2^n$ for $d \geq n$. In general, $C(n, d)$ is given by*

$$C(n, d) = C(n-1, d-1) + C(n-1, d) = 2 \sum_{i=0}^{d-1} \binom{n-1}{i}. \quad (5.3)$$

Proof. The number of intersected orthants is exactly equal to the number of proper sign patterns (excluding zero values) that can be generated by linear combinations of those d vectors. When $d = 1$, there can only be two such sign patterns corresponding to positive and negative multiples of that vector, thus giving $C(n, 1) = 2$. Whenever $d \geq n$, we can choose a basis for \mathbb{R}^n and add additional vectors as needed, and we can reach all points, and therefore all $2^n = C(n, d)$ sign patterns.

For the general case (5.3), let v_1, \dots, v_d be vectors in \mathbb{R}^n such that the affine hull with the origin, $S = \text{aff}\{0, v_1, \dots, v_d\}$, gives a subspace in \mathbb{R}^n that properly intersects the maximum number of orthants, $C(n, d)$. Without loss of generality assume that vectors v_i , $i = 1, \dots, d-1$ all have their n th component equal to zero. Now, let $T = \text{aff}\{0, v_1, \dots, v_{d-1}\} \subseteq \mathbb{R}^{n-1}$ be the intersection of S with the $(n-1)$ -dimensional subspace of all points $\mathcal{X} = \{x \in \mathbb{R}^n \mid x_n = 0\}$, and let C_T denote the number of $(n-1)$ -orthants intersected by T . Note that T itself, as embedded in \mathbb{R}^n , does not properly intersect any orthant. However, by adding or subtracting an arbitrarily small amount of v_d , we intersect $2C_T$ orthants; taking v_d to be the n th column of the identity matrix would suffice for that matter. Any other orthants that are added have either $x_n > 0$ or $x_n < 0$, and their number does not depend on the magnitude of the n th entry of v_d , provided it remains nonzero. Because only the first $n-1$ entries of v_d determine the maximum number of additional orthants, the problem reduces to \mathbb{R}^{n-1} . In fact, we ask how many new orthants can be added to C_T taking the affine hull of T with v , the orthogonal projection v_d onto \mathcal{X} . Since the maximum orthants for

¹Henceforth we use the convention that the uniqueness of a sign pattern is invariant under negation.

this d -dimensional subspace in \mathbb{R}^{n-1} is given by $C(n-1, d)$, this number is clearly bounded by $C(n-1, d) - C_T$. Adding this to $2C_T$, we have

$$\begin{aligned} C(n, d) &\leq 2C_T + [C(n-1, d) - C_T] = C_T + C(n-1, d) \\ &\leq C(n-1, d-1) + C(n-1, d) \\ &\leq 2 \sum_{i=0}^{d-1} \binom{n-1}{i}. \end{aligned} \tag{5.4}$$

The final expression follows by expanding the recurrence relations, which generates (a part of) Pascal's triangle, and combining this with $C(1, j) = 2$ for $j \geq 1$. In the above, whenever there are free orthants in \mathbb{R}^{n-1} , that is, when $d < n$, we can always choose the corresponding part of v_d in that orthant. As a consequence we have that no subspace generated by a set of vectors can intersect the maximum number of orthants when the range of those vectors includes some e_i .

We now show that this expression holds with equality. Let U denote an $(n-d)$ -subspace in \mathbb{R}^n that intersects the maximum $C(n, n-d)$ orthants. We now claim that in the interior of each orthant not intersected by U there exists a vector that is orthogonal to U . If this were not the case then T must be aligned with some e_i and can therefore not be optimal. The span of these orthogonal vectors generates a d -subspace V that intersects $C_V = 2^n - C(n, n-d)$ orthants, and it follows that

$$\begin{aligned} C(n, d) &\geq C_V = 2^n - C(n, n-d) \\ &\geq 2^n - 2 \sum_{i=0}^{n-d-1} \binom{n-1}{i} = 2 \sum_{i=0}^{n-1} \binom{n-1}{i} - 2 \sum_{i=0}^{n-d-1} \binom{n-1}{i} \\ &= 2 \sum_{i=n-d}^{n-1} \binom{n-1}{i} = 2 \sum_{i=0}^{d-1} \binom{n-1}{i} \geq C(n, d), \end{aligned}$$

where the last inequality follows from (5.4). Consequently, all inequalities hold with equality. \square

Corollary 5.2. *Given $d \leq n$, then $C(n, d) = 2^n - C(n, n-d)$, and $C(2d, d) = 2^{2d-1}$.*

Corollary 5.3. *A subspace \mathcal{H} in \mathbb{R}^n , defined as the range of $V = [v_1, v_2, \dots, v_d]$, intersects the maximum number of orthants $C(n, d)$ whenever $\text{rank}(V) = n$, or when $e_i \notin \text{range}(V)$ for $i = 1, \dots, n$.*

5.2 Practical considerations

In practice it is generally not feasible to generate all of the $C(|\mathcal{I}|, r)/2$ unique sign patterns. This means that we would have to set S in (5.2) to the number of unique patterns actually tried. For a given X_0 the actual probability of recovery is determined by a number of factors. First of all, the linear combinations of the columns of the nonzero part of \bar{X} prescribe a subspace and therefore a set of possible sign patterns. With each sign pattern is associated a face in \mathcal{C} that may or may not map to a face in AC . In addition, depending on the probability distribution from which the weight vectors w are drawn, there is a certain probability for reaching each sign pattern. Summing the probability of reaching those patterns that can be recovered gives the probability $P(A, \mathcal{I}, X_0)$ of recovering with an individual random sample w . The probability of recovery after t trials is then of the form

$$1 - [1 - P(A, \mathcal{I}, X_0)]^t.$$

To attain a certain sign pattern \bar{e} , we need to find an r -vector w such that $\text{sign}(\bar{X}w) = \bar{e}$. For a positive sign on the j th position of the support we can take any vector w in the open halfspace $\{w \mid \bar{X}^{j-} w > 0\}$, and likewise for negative signs. The region of vectors w in \mathbb{R}^r that generates a desired sign pattern thus corresponds to the intersection of $|\mathcal{I}|$ open halfspaces. The measure of this intersection as a fraction of \mathbb{R}^r determines the probability of sampling such a w . To formalize,

define \mathcal{K} as the cone generated by the rows of $-\text{diag}(\bar{e})\bar{X}$, and the unit Euclidean $(r-1)$ -sphere $\mathcal{S}^{r-1} = \{x \in \mathbb{R}^r \mid \|x\|_2 = 1\}$. The intersection of halfspaces then corresponds to the interior of the polar cone of \mathcal{K} : $\mathcal{K}^\circ = \{x \in \mathbb{R}^r \mid x^T y \leq 0, \forall y \in \mathcal{K}\}$. The fraction of \mathbb{R}^r taken up by \mathcal{K}° is given by the $(r-1)$ -content of $\mathcal{S}^{r-1} \cap \mathcal{K}^\circ$ to the $(r-1)$ -content of \mathcal{S}^{r-1} [21]. This quantity coincides precisely with the definition of the external angle of \mathcal{K} at the origin.

5.3 Experiments

In this section we illustrate the results from Section 5 and examine some practical considerations that affect the performance of ReMBo- ℓ_1 . For all experiments that require the matrix A , we use the same 20×80 matrix that was used in Section 4, and likewise for the supports \mathcal{I}_s . To solve (1.2), we again use CVX in conjunction with SDPT3. We consider x_0 to be recovered from $b = Ax_0 = AX_0 w$ if $\|x^* - x_0\|_\infty \leq 10^{-5}$, where x^* is the computed solution.

The experiments that are concerned with the number of unique sign patterns generated depend only on the $s \times r$ matrix \bar{X} representing the nonzero entries of X_0 . Because an initial reordering of the rows does not affect the number of patterns, those experiments depend only on \bar{X} , $s = |\mathcal{I}|$, and the number of observations r ; the exact indices in the support set \mathcal{I} are irrelevant for those tests.

5.3.1 Generation of unique sign patterns

The practical performance of ReMBo- ℓ_1 depends on its ability to generate as many different sign patterns using the columns in X_0 as possible. A natural question to ask then is how the number of such patterns grows with the number of randomly drawn samples w . Although this ultimately depends on the distribution used for generating the entries in w , we shall, for sake of simplicity, consider only samples drawn from the normal distribution. As an experiment we take a 10×5 matrix \bar{X} with normally-distributed entries, and over 10^8 trials record how often each sign-pattern (or negation) was reached, and in which trial they were first encountered. The results of this experiment are summarized in Figure 7. From the distribution in Figure 7(b) it is clear that the occurrence levels of different orthants exhibits a strong bias. The most frequently visited orthant pairs were reached up to 7.3×10^6 times, while others, those hard to reach using weights from the normal distribution, were observed only four times over all trials. The efficiency of ReMBo- ℓ_1 depends on the rate of encountering new sign patterns. Figure 7(c) shows how the average rate changes over the number of trials. The curves in Figure 7(d) illustrate the probability given in (5.2), with S set to the number of orthant pairs at a given iteration, and with face counts determined as in Section 4, for three instances with support cardinality $s = 10$, and observations $r = 5$.

5.3.2 Role of \bar{X} .

Although the number of orthants that a subspace can intersect does not depend on the basis with which it was generated, this choice does greatly influence the ability to sample those orthants. Figure 8 shows two ways in which this can happen. In part (a) we sampled the number of unique sign patterns for two different 9×5 matrices \bar{X} , each with columns scaled to unit ℓ_2 -norm. The entries of the first matrix were independently drawn from the normal distribution, while those in the second were generated by repeating a single column drawn likewise and adding small random perturbations to each entry. This caused the average angle between any pair of columns to decrease from 65 degrees in the random matrix to a mere 8 in the perturbed matrix, and greatly reduces the probability of reaching certain orthants. The same idea applies to the case where $d \geq n$, as shown in part (b) of the same figure. Although choosing d greater than n does not increase the number of orthants that can be reached, it does make reaching them easier, thus allowing ReMBo- ℓ_1 to work more efficiently. Hence, we can expect ReMBo- ℓ_1 to have higher recovery on average when the number of columns in X_0 increases and when they have a lower mutual coherence $\mu(X) = \min_{i \neq j} |x_i^T x_j| / (\|x_i\|_2 \cdot \|x_j\|_2)$.

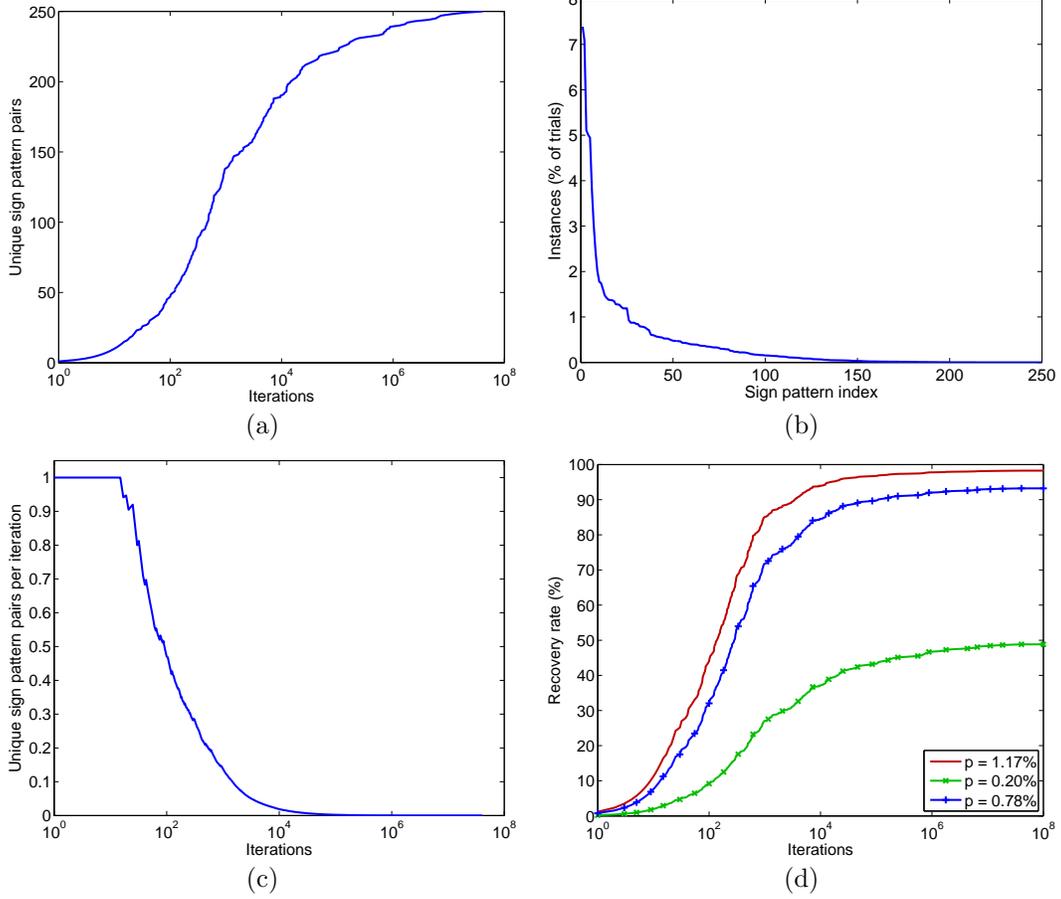


Figure 7: Sampling the sign patterns for a 10×5 matrix \bar{X} , with (a) number of unique sign patterns versus number of trials, (b) relative frequency with which each orthant is sampled, (c) average number of new sign patterns per iteration as a function of iterations, and (d) modeled probability of recovery using ReMBo- ℓ_1 for three instances of X_0 with row sparsity $s = 10$, and $r = 5$ observations.

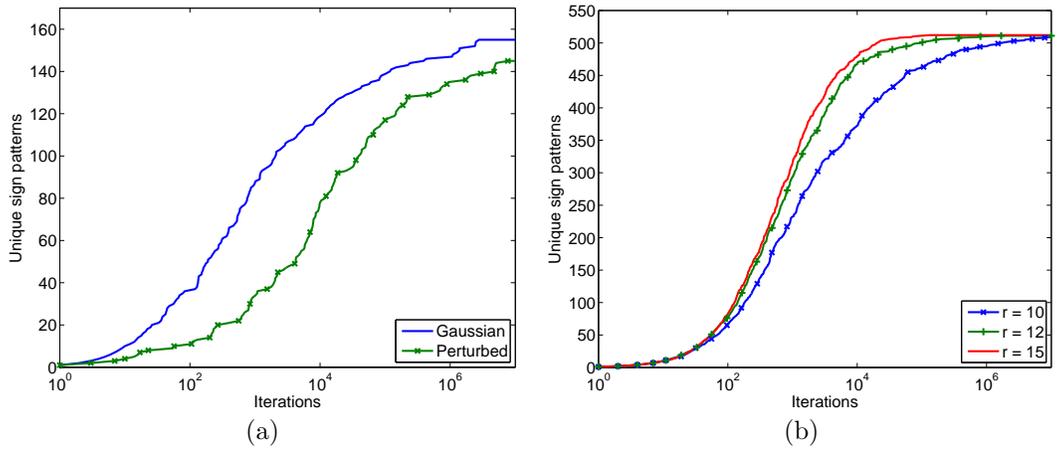


Figure 8: Number of unique sign patterns for (a) two 9×5 matrices \bar{X} with columns scaled to unit ℓ_2 -norm; one with entries drawn independently from the normal distribution, and one with a single random column repeated and random perturbations added, and (b) $10 \times r$ matrices with $r = 10, 12, 15$.

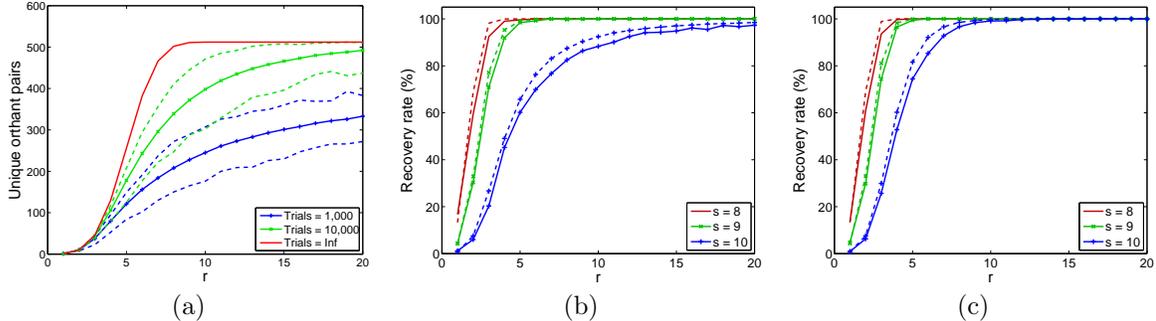


Figure 9: Effect of limiting the number of weight vectors w on (a) the distribution of unique orthant counts for $10 \times k$ random matrices \bar{X} , solid lines give the median number and the dashed lines indicate the minimum and maximum values, the top solid line is the theoretical maximum; (b–c) the average performance of the ReMBo- ℓ_1 algorithm (solid) for fixed 20×80 matrix A and three different support sizes $r = 8, 9, 10$, along with the average predicted performance (dashed). The support patterns used are the same as those used for Figure 4.

5.3.3 Limiting the number of iterations

The number of iterations used in the previous experiments greatly exceeds that which is practically feasible: we cannot afford to run ReMBo- ℓ_1 until all possible sign patterns have been tried, even if there was a way to detect that the limit had been reached. Realistically, we should set the number of iterations to a fixed maximum that depends on the computational resources available, and the problem setting itself.

In Figure 7 we show the empirical number of unique orthants S as a function of iterations along with the corresponding recovery rate from (5.2). When using only a limited number of iterations it is interesting to know what the distribution of unique orthant counts looks like. To find out, we drew 1,000 random \bar{X} matrices for each size $s \times r$, with $s = 10$ nonzero rows fixed, and the number of columns ranging from $r = 1, \dots, 20$. For each \bar{X} we counted the number of unique sign patterns attained after respectively 1,000 and 10,000 iterations. The resulting minimum, maximum, and median values are plotted in Figure 9(a) along with the theoretical maximum. More interestingly of course is the average recovery rate of ReMBo- ℓ_1 with those number of iterations. For this test we again used the 20×80 matrix A with predetermined support \mathcal{I} , and with success or failure of each sign pattern on that support precomputed. For each value of $r = 1, \dots, 20$ we generated random matrices X on \mathcal{I} and ran ReMBo- ℓ_1 with the maximum number of iterations set to 1,000 and 10,000. To save on computing time, we compared the on-support sign pattern of each combined coefficient vector Xw to the known results instead of solving ℓ_1 . The average recovery rate thus obtained is plotted in Figures 9(b)–(c), along with the average of the modeled performance using (5.2) with S set to the orthant counts found in the previous experiment.

6 Conclusions

The MMV problem is often solved by minimizing the sum-of-row norms of the unknown coefficients X . We show that the (local) uniform recovery properties, i.e., recovery of all X_0 with a fixed row support $\mathcal{I} := \text{Supp}_{\text{row}}(X_0)$, cannot exceed that of $\ell_{1,1}$, the sum of ℓ_1 norms. This is despite the fact that $\ell_{1,1}$ reduces to solving the basis pursuit problem (1.2) for each column separately, which does not take advantage of the fact that all vectors in X_0 are assumed to have the same support. A consequence of this observation is that the use of restricted isometry techniques to analyze (local) uniform recovery using sum-of-norm minimization can at best give improved bounds on ℓ_1 recovery.

Empirically, minimization with $\ell_{1,2}$, the sum of ℓ_2 norms, clearly outperforms $\ell_{1,1}$ on individual problem instances: for supports where uniform recovery fails, $\ell_{1,2}$ recovers more cases than $\ell_{1,1}$. We construct cases where $\ell_{1,2}$ succeeds while $\ell_{1,1}$ fails, and vice versa. From the construction where only $\ell_{1,2}$ succeeds it also follows that the relative magnitudes of the coefficients in X_0 matter for

recovery. This is unlike $\ell_{1,1}$ recovery, where only the support and the sign patterns matter. This implies that the notion of faces, so useful in the analysis of ℓ_1 , disappears.

We show that the performance of $\ell_{1,1}$ outside the uniform-recovery regime degrades rapidly as the number of observations increases. This situation can be turned around by using the ReMBo method [23] which reduces the problem to a series of SMV problems followed by boosting once a promising candidate support is found. In the setting where ReMBo is used with an ℓ_1 -subproblem solver, it is important that the reduction stage generates as many different sign patterns in the mixed vectors as possible, in order to increase the potential recovery rate. We give a tight upper bound on the number of sign patterns that can be reached in terms of the number of observations in X_0 and the cardinality of the joint support.

Based on the geometrical interpretation of ReMBo- ℓ_1 (cf. Figure 5), we conclude that the theoretical bound on performance does not increase with the number of observations after this number reaches the number linearly independent nonzero rows in X_0 . In practice such an increase does matter because it aids the sampling of sign patterns.

In addition we develop a simplified model for the performance of ReMBo- ℓ_1 . To improve the model we would need to know the distribution of faces in the cross-polytope \mathcal{C} that map to faces on AC , and the distribution of external angles for the cones generated by the signed rows of the nonzero part of X_0 . To predict the practical performance we would also need to know how the number of unique sign patterns in the mixed vectors develops as a function of the number of reduction trials. Both topics remain as future work.

Finally, it would be very interesting to have a rigorous comparison between the recovery performance of $\ell_{1,2}$ and ReMBo- ℓ_1 . At present the only available comparison is given by the preliminary findings by Eldar and Mishali [13].

All of the numerical experiments in this paper are reproducible. The scripts used to run the experiments and generate the figures can be downloaded from

<http://www.cs.ubc.ca/~mpf/jointsparse>.

Acknowledgments

The authors would like to give their sincere thanks to Özgür Yılmaz and Rayan Saab for their thoughtful comments and suggestions during numerous discussions, and thanks to two anonymous referees for numerous and detailed suggestions that spurred significant improvements to this paper.

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