Polar Deconvolution of Mixed Signals

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Abstract. The signal demixing problem seeks to separate a superposition of multiple signals into its constituent components. This paper describes a two-stage approach that first decompresses and subsequently deconvolves the noisy and undersampled observations of the superposition. Probabilistic error bounds are given on the accuracy with which this process approximates the individual signals. The theory of polar convolution of convex sets and gauge functions plays a central role in the analysis and solution process. If the measurements are random and the noise is bounded, this approach stably recovers low-complexity and mutually incoherent signals, with high probability and with optimal sample complexity. We develop an efficient algorithm, based on level-set and conditional-gradient methods, that solves the convex optimization problems with sublinear iteration complexity and linear space requirements. Numerical experiments on both real and synthetic data confirm the theory and the efficiency of the approach.

Key words. signal demixing, polar convolution, atomic sparsity, convex optimization

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1. Introduction. The signal demixing problem seeks to separate a superposition of signals into its constituent components. In the measurement model we consider, a set of signals \( \{x_i^\natural\}_{i=1}^k \) in \( \mathbb{R}^n \) are observed through noisy measurements \( b \in \mathbb{R}^m \), with \( m \leq n \), of the form

\[
b = M x_S^\natural + \eta \quad \text{with} \quad x_S^\natural := \sum_{i=1}^k x_i^\natural.
\]

The known linear operator \( M : \mathbb{R}^n \to \mathbb{R}^m \) models the acquisition process of the superposition vector \( x_S^\natural \). The vector \( \eta \in \mathbb{R}^m \) represents noise uncorrelated with the data. This measurement model and its variations are useful for a range of data-science applications, including mixture models [3, 31], blind deconvolution [1], blind source separation [9], and morphological component analysis [6].

A central concern of the demixing problem is to delineate efficient procedures and accompanying conditions that make it possible to recover the constituent signals to within a prescribed accuracy—using the fewest number of measurements \( m \). The recovery of signals from the measurements modeled in (1.1) cannot be accomplished without additional information, such as the latent structure in each signal \( x_i^\natural \). We build on the general atomic-sparsity framework formalized by Chandrasekaran et al. [10], and assume that each signal \( x_i^\natural \) is itself well represented as a superposition of a few atomic signals from a collection \( A_i \subset \mathbb{R}^n \). In other words, the vectors \( \{x_i^\natural\}_{i=1}^k \) are paired with atomic sets \( \{A_i\}_{i=1}^k \) that allow the decompositions

\[
x_i^\natural = \sum_{a \in A_i} c_a a, \quad c_a \geq 0 \quad \forall a \in A_i,
\]
where most of the coefficients $c_a$ are zero.

This model of atomic sparsity includes a range of important notions of sparsity, such as sparse vectors, which are sparse in the set of canonical vectors, and low-rank matrices, which are sparse in the set of rank-1 matrices with unit spectral norm. Other important generalizations include higher-order tensor decompositions, useful in computer vision [40] and handwritten digit classification [33], and polynomial atomic decomposition [8].

We follow a line of analysis championed by McCoy and Tropp [25,26], Amelunxen et al. [2], and Oymak and Tropp [29], who developed a suite of geometric tools based on the statistical dimension of convex cones to measure the probability that the cones intersect under random rotations. These papers describe conditions under which the solution of a single convex optimization problem yields or approximates each of the individual signals $x_i$. As described by Amelunxen et al. [2], the statistical dimension of a cone shares many properties with its Gaussian width, which is a related concept used by Chandrasekaran et al. [11] in their derivation of conditions for recovering a single compressively-sampled signal.

The gauge to an atomic set $A$, defined by the convex function

$$
\gamma_A(x) := \inf_{c_a} \left\{ \sum_{a \in A} c_a \mid x = \sum_{a \in A} c_a a, \ c_a \geq 0 \ \forall a \in A \right\},
$$

is central to the formulation of a convex optimization process that provably leads to solutions that have sparse decompositions in the sense of (1.2). The properties of gauges and their relationship with atomic sparsity are outlined by Chandrasekaran et al. [10] and Fan et al. [15].

1.1. Polar convolution. The typical approach to the demixing problem is to combine $k$ separate gauge functions, each corresponding to one of the atomic sets $\{A_i\}_{i=1}^k$, as a weighted sum or similar formulation. We instead combine the separate gauge functions using a special-purpose infimal convolution operation that reflects the additive structure of the superposition, as defined in (1.1).

For any two atomic sets $A_1$ and $A_2$, the polar convolution of the corresponding gauge functions is

$$
(\gamma_{A_1} \circ \gamma_{A_2})(x) := \inf_{x_1,x_2} \max \left\{ \gamma_{A_1}(x_1), \ \gamma_{A_2}(x_2) \mid x = x_1 + x_2 \right\}.
$$

The resulting function is the gauge to the vector sum $A_1 + A_2$, i.e.,

$$
\gamma_{A_1} \circ \gamma_{A_2} = \gamma_{A_1 + A_2};
$$

see [19, Proposition 6.2]. This convolution operation was first described by Rockafellar [32, Theorem 5.8] as a convexity-preserving operation for general convex functions. Friedlander et al. [19] specialized this operation to the family of gauge functions and describe significant properties that accrue when the convolution is used in that context.

The subdifferential properties of this convolution facilitate our analysis and allow us to build an efficient algorithm that is practical for a range of problems. In particular, the polar convolution decouples under a duality correspondence built around the polarity of convex sets. The polar to a convex set $C \subset \mathbb{R}^n$,

$$
C^\circ = \{ y \in \mathbb{R}^n \mid \langle x, y \rangle \leq 1 \text{ for all } x \in C \},
$$
contains a dual description of $C$ in terms of all of its supporting hyperplanes. Under this dual representation,

$$\gamma(A_1 + A_2)^o = \gamma A_1^o + \gamma A_2^o,$$

which implies that the subdifferential decouples as $\partial \gamma(A_1 + A_2)^o = \partial \gamma A_1^o + \partial \gamma A_2^o$. Thus, a subgradient computation, which is central to all first-order methods for convex optimization, can be implemented using only subdifferential oracles for each of the polar functions $\gamma A_i^o$. We show in section 5 how to use this property to implement a version of the conditional gradient method [18,21] to obtain the polar decomposition using space complexity that scales linearly with the size of the data.

1.2. Decompression and deconvolution. The principle innovation of our approach to the demixing problem is to decouple the recovery procedure into two stages: an initial decompression stage meant to recover the superposition $x_S^\natural$ from the vector of observations $b$, followed by a deconvolution stage that separates the recovered superposition $x_S^\natural$ into its constituent components $\{x_i^\natural\}_{i=1}^k$. We couple the convex theory of polar convolution [19] to the theory of statistical dimension and signal incoherence to derive a recovery procedure and analysis for demixing a compressively sampled mixture to within a prescribed accuracy.

Stage 1: Decompression. The initial decompression stage is based on the observation that because each signal $x_i^\natural$ is $A_i$ sparse, the superposition $x_S^\natural$ must be sparse with respect to the weighted vector sum

$$A_S := \sum_{i=1}^k \lambda_i A_i \equiv \\left\{ \sum_{i=1}^k \lambda_i a_i \mid a_i \in A_i \cup \{0\}, \ i \in 1:k \right\}$$

of the individual atomic sets $A_i$. The positive weights $\lambda_i$ carry information about the relative powers of the individual signals, and serve to equilibrate the gauge values of each signal. Thus, the weights $\lambda_i$ are defined so that for each $i \in 1:k$,

$$\gamma_{\lambda_i A_i}(x_i^\natural) = \gamma_{A_i}(x_i^\natural).$$
The initial decompression stage solves the convex optimization problem

\[(P1) \minimize_{x \in \mathbb{R}^n} \gamma_{A_i}(x) \text{ subject to } \|Mx - b\| \leq \alpha,\]

where the parameter \(\alpha \geq 0\) bounds the acceptable level of misfit between the linear model \(Mx\) and the observations \(b\), and correspondingly reflects the anticipated magnitude of the noise \(\eta\).

Proposition 3.2 establishes conditions under which the solution \(x^*_S\) to \((P1)\) coincides exactly, or stably approximates, the superposition \(x^\natural_S\). It follows from (1.4) that the objective of \((P1)\) is in fact the polar convolution of the individual weighted gauges:

\[\gamma_{A_S}(x) = \gamma_{\lambda_1 A_1} \odot \gamma_{\lambda_2 A_2} \odot \cdots \odot \gamma_{\lambda_k A_k}(x).\]

**Stage 2: Deconvolution.** The solution \(x^*_S\) of the decompression problem \((P1)\) defines the subsequent convex deconvolution problem

\[(P2) \minimize_{x_1,\ldots,x_k} \max_{i \in [k]} \gamma_{\lambda_i A_i}(x_i) \text{ subject to } \sum_{i=1}^k x_i = x^*_S\]

to obtain approximations \(x^*_i\) to each constituent signal \(x^\natural_i\). This problem effectively inverts the polar convolution operation defined by (1.3).

In both stages, a variant of the conditional-gradient method provides a computationally and memory efficient algorithm that can be implemented with storage proportional to the number of measurements \(m\). We describe in section 5 the details of the method.

**1.3. Prior work.** The history of signal demixing can be traced to early work in seismic imaging [12] and morphological component analysis [6, 34], which used 1-norm regularization to separate noisy signals. More recently, McCoy and Tropp [25, 26] and Oymak and Tropp [29] proposed a unified theoretical framework for signal demixing using modern tools from high-dimensional geometry.

McCoy and Tropp [26] analyzed the recovery guarantees of a convex program meant to reconstruct \(k = 2\) randomly-rotated signals from a full set of noiseless observations, i.e., \(m = n\) and \(\|\eta\| = 0\). McCoy and Tropp [25] subsequently extended this framework to demixing \(k \geq 2\) randomly-rotated signals from noisy measurements, as modeled by (1.1). Oymak and Tropp [29] consider a demixing problem similar to \((P2)\) that also incorporates the measurement operator \(M\), and provide guarantees for demixing two unknown vectors from random and noiseless measurements. We build on this line of work by providing explicit recovery error bounds in terms of the complexity of the signal sets and the number of observations. The analysis allows for any number of individual signals \(k \geq 2\). We postpone to section 4 a detailed comparison between our results and earlier work.

Early work on demixing sparse signals implicitly assumed some notion of incoherence between representations of the signals. This concept was made concrete by Donoho and Huo [14], and subsequently Donoho and Elad [13], who measured the mutual incoherence of finite bases via the maximal inner-products between elements of the sets. Related incoherence definitions appear in compressed sensing [23, 35] and robust PCA [7, 42]. In this paper we adopt McCoy and Tropp’s [25] notion of incoherence as the minimal angle between conic representation of the individual signals.
Figure 2. A non-trivial intersection of $D(A,x^*)$ and $\text{null}(M)$ is required for successful decompression. The blue shaded region represents the shifted descent cone $x^* + D(A,x^*)$, and red line represents the shifted null space $\text{null}(M) + x^*$. If $D(A,x^*) \cap \text{null}(M) \neq \{0\}$ (as depicted here) then there exists a vector $\hat{x}$ such that $\gamma_A(\hat{x}) < \gamma_A(x^*)$ and $M\hat{x} = Mx^*$.

1.4. Contributions and roadmap. Section 2 develops a bound on the error with which a solution $x^*_S$ of the decompression problem (P1) approximates $x^*_S$. Proposition 2.2 characterizes the error in terms of the overall complexity of the signal. This result flows from Tropp [36]. We rely on a conic decomposition property particular to polar convolution.

Section 3 develops a bound on the error with which solutions of the deconvolution problem (P2) approximate each $x^*_i$. This bound is given in terms of the error in the initial decompression process and the incoherence between signals as measured by the minimum angle between certain conic representations of each signal; see Proposition 3.2. This result requires a general notion of incoherence based on the angle between descent cones, first analyzed by McCoy and Tropp [25].

Section 4 shows how a random-rotation model yields a particular level of incoherence with high probability; see Proposition 4.3. We develop the recovery guarantee under the random-rotation model; see Theorem 4.4. These results are verified numerically in section 6.1.

Section 5 outlines an algorithm based on conditional-gradient and level-set methods for computing the decompression and deconvolution process. The worst-case computational complexity of this process is sublinear in the required accuracy. Numerical experiments in section 6 on real and synthetic structured signals verify the efficiency of the approach.

The following blanket assumption on the data holds throughout the paper.

**Assumption 1.1 (Measurement model).** The linear model (1.1) satisfies the following conditions: the linear map $M : \mathbb{R}^n \to \mathbb{R}^m$ has i.i.d. standard Gaussian entries; the noise vector $\eta$ satisfies $\|\eta\| \leq \alpha$ for some known scalar $\alpha$; and the relative signal powers $\{\lambda_i\}_{i=1}^k$ satisfy (1.6).

The proofs of all mathematical results are given in section 7. Throughout this paper, the 2-norm of a vector $x$ is denoted by $\|x\| = \sqrt{\langle x, x \rangle}$, unless another norm is specified.
2. Decompressing the superposition. We argued in Subsection 1.2 that under the assumption that the individual signals \( x^i \) are \( A_i \)-sparse, the aggregate signal \( x^S \) is sparse with respect to the aggregate atomic set \( A_S \). Thus, the decompression of the observation \( b \) in (1.1) is accomplished by minimizing the gauge to \( A_S \) to within the bound on the noise \( \| \eta \| \leq \alpha \), as modeled by the recovery problem (P1). Without noise (i.e., \( \alpha = 0 \)), the signal \( x^S \) is the unique solution to this problem when the null space of the measurement operator \( M \) has only a trivial intersection with the descent cone

\[
D_S := D(A_S, x^S) = \text{cone} \{ d \in \mathbb{R}^n \mid \gamma_{A_S}(x^S + d) \leq \gamma_{A_S}(x^S) \}.
\]

where \( \text{cone}C := \mathbb{R}_+C \) is the conic extension of a set \( C \). In other words, \( x^S \) is the unique solution of (P1) if and only if

\[
D_S \cap \text{null}(M) = \{0\}.
\]

Figure 2 illustrates the geometry of this optimality condition, and depicts a case in which it doesn’t hold.

If the linear operator \( M \) is derived from Gaussian measurements, Gordon [20] characterizes the probability of the event (2.1) as a function of the Gaussian width of the descent cone \( D_S \) and the number of measurements \( m \). This result forms the basis for recovery guarantees developed by Chandrasekaran et al. [10] and Tropp [36] for a related convex formulation.

Intuitively, the number of measurements required for stable recovery of the aggregate \( x^S \) depends on the total complexity of the \( k \) constituent \( A_i \)-sparse vectors \( x^i \). The complexity is measured in terms of the statistical dimension of each of the descent cones \( D_i \): for any convex cone \( D \subset \mathbb{R}^n \), its statistical dimension is given by

\[
\delta(D) = \mathbb{E} \left[ \| \text{proj}_D(g) \|^2 \right],
\]

where \( g \) is standard normal random vector and \( \text{proj}_D \) is the orthogonal projection onto the cone \( D \). Tropp [36, Corollary 3.5] establishes a bound on the error between the solutions of the decompression problem (P1) and the superposition \( x^S \) that depends on the statistical dimension \( \delta(D_S) \) of its descent cone. The following lemma asserts that this quantity cannot be greater than the sum of the statistical dimensions \( \delta(D_i) \) for each signal \( x^i \).

Lemma 2.1. The statistical dimensions of the descent cones \( D_S \) and \( \{D_i\}^k_{i=1} \) satisfy

\[
\Delta := \sum_{i=1}^k \delta(D_i) \geq \delta(D_S).
\]

The following proposition is a restatement of Tropp [36, Corollary 3.5] that uses the sum of statistical dimensions of all of the individual signals. Let \( [\xi]_+ = \max\{0, \xi\} \).

Proposition 2.2 (Stable decompression of the aggregate). For any \( t > 0 \), any solution \( x^* \) of (P1) satisfies

\[
\| x^* - x^S \| \leq 2 \alpha \left[ \sqrt{m - 1} - \sqrt{\Delta - t} \right]_+^{-1}
\]

with probability at least \( 1 - \exp(-t^2/2) \).
3. Deconvolving the components. The deconvolution problem (P2) can only succeed in separating the superposition $x^S$ into its components $\{x^i\}_{i=1}^k$ if each of these has significant differences in their atomic representations. As counterexamples, it’s not possible to separate the superposition of two sparse signals or two low-rank signals without additional information. We follow McCoy and Tropp [25], and measure the dissimilarity between signal structures—and thus their incoherence—using the angles between corresponding descent cones.

To motivate the incoherence definition, consider the case where there are only $k = 2$ signals $x^1$ and $x^2$. If the descent cones $-D_1$ and $D_2$ have a nontrivial intersection, then there exists a nonzero direction $d \in -D_1 \cap D_2$ such that $\gamma_{A_1}(x^1 - d) < \gamma_{A_1}(x^1)$ and $\gamma_{A_2}(x^2 + d) < \gamma_{A_2}(x^2)$, which contradicts the optimality required for $x^1$ and $x^2$. Thus, deconvolution cannot succeed unless the descent cones intersect only at the origin. Figure 3 illustrates this geometry.

Obert [27] defines the angle between two cones $K_1$ and $K_2$ in $\mathbb{R}^n$ as the minimal angle between vectors in these two cones. It follows that the cosine of the angle between two cones...
can be expressed as
\[
\cos \angle (\mathcal{K}_1, \mathcal{K}_2) = \sup \{ \langle x, y \rangle \mid x \in \mathcal{K}_1 \cap S^{n-1}, y \in \mathcal{K}_2 \cap S^{n-1} \}.
\]

For the general case where the number of signals \( k \geq 2 \), a natural choice for measure of incoherence is the minimum angle between the descent cone of a structured vector with respect to the remaining descent cones.

**Definition 3.1.** The pairs \( \{(x_i^1, A_i)\}_{i=1}^k \) are \( \beta \)-incoherent with \( \beta \in (0, 1] \) if for all \( i \in 1:k \),
\[
\cos \angle (-D_i, \sum_{j \neq i} D_j) \leq 1 - \beta.
\]

We use the incoherence between descent cones to bound the error between the true constituent signals \( x_i^S \) and the solution set of the deconvolution problem (P2). This bound depends on the accuracy of the approximation \( x_i^S \) to the true superposition \( x_i^S \).

**Proposition 3.2** (Stable deconvolution). If the pairs \( \{(x_i^1, A_i)\}_{i=1}^k \) are \( \beta \)-incoherent for some \( \beta \in (0, 1] \), then any set of solutions \( \{x_i^*\}_{i=1}^k \) of (P2) satisfies for all \( i \in 1:k \)
\[
\|x_i^* - x_i^S\| \leq \|x_i^S - x_i^S\|/\sqrt{\beta},
\]
where \( x_i^S \) is any solution of (P1).

In summary, a large angle between a descent cone \( D_i \) and all the other descent cones—as reflected by a large incoherence constant \( \beta \)—corresponds a small error between each \( x_i^* \) and the ground truth \( x_i^S \).

4. **Inducing incoherence through random rotation.** Proposition 3.2 establishes the accuracy of the deconvolution problem using the \( \beta \)-incoherence between signals, as formalized by Definition 3.1. Except in very special cases, however, it’s not feasible to determine the incoherence constant. We build on McCoy and Tropp’s random rotation model [25] to quantify with high probability the \( \beta \)-incoherence of \( k \) randomly-rotated atomic sparse signals.

Let’s start with something simpler. The following proposition provides a probabilistic bound on the angle between two cones, with one of them randomly rotated, based on their statistical dimensions. This geometric result maybe of intrinsic interest in other contexts. Let \( \text{SO}(n) \) denote the special orthogonal group, which consists of all \( n \)-by-\( n \) orthogonal matrices with unit determinant.

**Proposition 4.1** (Probabilistic bound on angle under random rotation). Let \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) be two closed convex cones in \( \mathbb{R}^n \). Then
\[
\mathbb{P} \left[ \cos \angle (Q\mathcal{K}_1, \mathcal{K}_2) \geq c \sqrt{\frac{\delta(\mathcal{K}_1) + \delta(\mathcal{K}_2)}{n-1}} + \frac{1}{2} \right] \leq \exp \left( -\frac{1}{16} (n-1) \right),
\]
where \( Q \) is drawn uniformly at random from \( \text{SO}(n) \) and \( c \) is a positive universal constant.

Now assume that the \( k \) signals \( x_i^S \) and their corresponding atomic sets \( A_i \) are defined via random rotations of some underlying signals and their corresponding atomic sets.
**Assumption 4.2 (Random rotations).** For each \(i \in 1:k\),

\[
x_{i}^\natural := Q_{i} \hat{x}_{i}^\natural \quad \text{and} \quad A_{i} := Q_{i} \hat{A}_{i},
\]

where each \(\hat{x}_{i}^\natural\) and \(\hat{A}_{i}\) is fixed, and the matrices \(Q_{i}\) are drawn uniformly and i.i.d. from \(\text{SO}(n)\).

The following corollary to Proposition 4.1 shows that under random rotation, the resulting signals are incoherent with high probability.

**Proposition 4.3.** Under Assumption 4.2, if \(\Delta < (n - 1)/\sqrt{c^2}\), where \(c\) is a positive universal constant, then the rotated pairs \(\{(x_{i}^\natural, A_{i})\}_{i=1}^{k}\) are \(\beta\)-incoherent with

\[
\beta := \frac{1}{2} - c\sqrt{\Delta/(n - 1)},
\]

with probability at least \(1 - k \exp(-\frac{1}{16}(n - 1))\).

We can now state the recovery guarantees for recovering randomly rotated vectors using the two-stage approach (P1) and (P2).

**Theorem 4.4.** Suppose that Assumptions 1.1 and 4.2 hold. If \(\Delta < (n - 1)/\sqrt{c^2}\), where \(c\) is a positive universal constant, then for any \(t > 0\), any set of minimizers \(\{x_{i}^*\}_{i=1}^{k}\) of (P2) satisfies for all \(i \in 1:k\)

\[
\|x_{i}^* - x_{i}^\natural\| \leq \frac{2\alpha}{\sqrt{\beta}} \left[\sqrt{m - 1} - \sqrt{\Delta - t}\right]^{-1}
\]

with probability at least \(1 - \exp(-t/2) - k \exp\left(-\frac{1}{16}(n - 1)\right)\), where \(\beta\) is given by (4.1).

The proof follows directly from Proposition 2.2, Proposition 3.2, Proposition 4.3, and the probability union bound. We verify empirically in section 6.1 the tightness of the bound (4.2).

**4.1. Comparison to McCoy and Tropp.** McCoy and Tropp [25] develop a demixing procedure based on the constrained optimization problem

\[
\min_{x_{1},...,x_{k}} \left\|M^{\dagger} \left(M \sum_{i=1}^{k} x_{i} - b\right)\right\| \quad \text{subject to} \quad \gamma(x_{i}) \leq \gamma_{A_{i}}(x_{i}^\natural), \quad \forall i \in 1:k,
\]

where \(M^{\dagger}\) is the Moore-Penrose pseudo-inverse of \(M\). They show that if \(n \geq m \geq \Delta + O(\sqrt{kn})\) and \(\{x_{i}^\natural\}_{i=1}^{k}\) are randomly rotated as per Assumption 4.2, then any set of minimizers \(\{x_{i}^*\}_{i=1}^{k}\) of (4.3) satisfies with high probability the bound

\[
\|x_{i}^* - x_{i}^\natural\| \leq C\|M^{\dagger}\eta\|
\]

for all \(i \in 1:k\) [25, Theorem A]. The constant \(C\) is unknown and may depend on all problem parameters except \(\eta\). As compared with this result, Theorem 4.4 offers two improvements. First, the bound in Theorem 4.4 makes explicit the effect of all problem parameters. Second, the bound on measurement complexity is tighter. In particular, Theorem 4.4 requires \(m \geq \Omega(\Delta)\), rather than \(m \geq \Delta + O(\sqrt{kn})\).
5. Deconvolution algorithm. We describe a procedure for obtaining solutions for the
decompression (P1) and deconvolution (P2) problems. The procedure first solves the decom-
pression problem (P1) using an algorithm that doesn’t store or track an approximation to
\( x^\#_S \), which in many contexts may be too large to store or manipulate directly. Instead, the
algorithm produces a sequence of iterates \( r^{(t)} := b - M x^{(t)} \) that approximate the residual
vector corresponding to an implicit approximation \( x^{(t)} \) of \( x^\#_S \). The procedure requires only the
storage of several vectors of length \( m \), which represents the size of the data \( b \). As we show
in section 5.3, the solution to the deconvolution problem (P2) is subsequently obtained via
an unconstrained linear least-squares problem that uses information implicit in this residual
vector. Algorithm 5.1 summarizes the overall procedure.

Algorithm 5.1 Decompression and deconvolution algorithm

Input: noise level \( \alpha > 0 \); accuracy \( \epsilon > 0 \)
1 \( \tau^0 \leftarrow 0 \)
2 for \( t \leftarrow 0, 1, 2, \ldots \) do [level-set iterations]
3 \( (r^{(t)}, p^{(t)}, \ell^{(t)}) \leftarrow \text{dual-conditional-gradient}(\tau^{(t)}) \) [solve (5.1) approximately]
4 if \( \|r^{(t)}\| > \sqrt{\alpha^2 + \epsilon} \) then break [test \( \epsilon \)-infeasibility for (P1); see (5.2)]
5 \( \tau^{(t+1)} \leftarrow \tau^{(t)} + (\ell^{(t)} - \alpha^2/2)/(\langle p^{(t)}, r^{(t)} \rangle) \) [Newton update to level-set parameter]
6 end
7 \( (x_1, \ldots, x_k) \leftarrow \text{solve (5.4)} \) using \( \bar{r} := r^{(t)} \) [solve deconvolution (P2)]
8 return \( (x_1, \ldots, x_k) \)

5.1. Level-set method. The loop beginning at Line 2 of Algorithm 5.1 approximately
solves a sequence of problems

\[
(5.1) \quad v(\tau) := \min_x \left\{ \frac{1}{2} \|M x - b\|^2 \mid \gamma_{A_x}(x) \leq \tau \right\},
\]

parameterized by the scalar \( \tau \) that defines the level-set constraint. This loop implements the
level-set approach \([4,38,39]\), which constructs a monotonically-increasing sequence \( \{\tau^{(t)}\} \) that
converges to the leftmost root \( \tau^* \) of the equation \( v(\tau) = \frac{1}{2} \alpha^2 \), which represents the fidelity
constraint of problem (P1). Under modest assumptions satisfied by this problem, the sequence
\( \tau^{(t)} \to \tau^* = \text{opt} \), the optimal value of (P1). The tail of the resulting sequence of computed
solutions to (5.1) is super-optimal and \( \epsilon \)-infeasible for (P1), i.e., a solution \( x \) satisfies

\[
(5.2) \quad \gamma_{A_x}(x) \leq \text{opt} \quad \text{and} \quad \|M x - b\| \leq \sqrt{\alpha^2 + \epsilon},
\]

where \( \epsilon \) is a specified optimality tolerance. The level-set algorithm requires \( O(\log(1/\epsilon)) \) ap-
proximate evaluations of the optimization problem (5.1) (see Line 3) to achieve this optimality
condition. Each approximate evaluation provides a global lower-minorant of \( v \) that is used by
a Newton-like update to the level-set parameter \( \tau^{(t)} \); see line 4.

5.2. Dual conditional gradient method. The level-set subproblems are solved approximate-
lly using the dual conditional-gradient method described by Algorithm 5.2. An implementa-
tion of this algorithm requires storage for three \( m \)-vectors

\[
p^{(t)} := M a^{(t)}, \quad q^{(t)} := M x^{(t)}, \quad r^{(t)} := b - M x^{(t)},
\]
Algorithm 5.2 dual-conditional-gradient(τ). This algorithm solves (P1) without reference to the primal iterate $x^{(t)}$, and instead returns the implied residual $r^{(t)} = b - Mx^{(t)}$.

Input: τ  
1 $r^{(0)} ← b$; $q^{(0)} ← 0$ 
2 for $t ← 0, 1, 2, \ldots$ do 
3   $p^{(t)} ∈ τF(MA_s; r^{(t)})$ \hspace{2cm} [\( p^{(t)} = Ma^{(t)} \) with \( a^{(t)} ∈ τF(MA_s; z^{(t)}) \); see (5.3)] 
4   $Δr^{(t)} ← p^{(t)} - q^{(t)}$ \hspace{2cm} [\( Δr^{(t)} = M(a^{(t)} - x^{(t)}) \)] 
5   $ρ^{(t)} ← \langle r^{(t)}, Δr^{(t)} \rangle$ \hspace{2cm} [optimality gap] 
6   if $ρ^{(t)} < \epsilon$ then break \hspace{2cm} [break if optimal] 
7   $θ^{(t)} ← \min \{ 1, ρ^{(t)}/\|Δr^{(t)}\|^2 \}$ \hspace{2cm} [exact linesearch on least-squares objective] 
8   $r^{(t+1)} ← r^{(t)} - θ^{(t)}Δr^{(t)}$ \hspace{2cm} [\( r^{(t+1)} = b - Mx^{(t+1)} \)] 
9   $q^{(t+1)} ← q^{(t)} + θ^{(t)}Δr^{(t)}$ \hspace{2cm} [\( q^{(t+1)} = Mx^{(t+1)} \)] 
10 end 
11 $ℓ^{(t)} ← \frac{1}{2}\|r^{(t)}\|^2 - ρ^{(t)}$ \hspace{2cm} [lower bound on optimal value] 
12 return $r^{(t)}$, $p^{(t)}$, $ℓ^{(t)}$

(A fourth vector $Δr^{(t)}$ can be computed at each iteration.) Implicit in these vectors are the iterate $x^{(t)}$ and current atom $a^{(t)} ∈ A_s$, which in some situations are prohibitively large to store or manipulate. The main computational cost is in Line 3, which uses the residual $r^{(t)}$ to expose an atom in the face

$$F(MA_s; r) = \text{conv} \{ p ∈ MA_s | \langle p, r \rangle = \sup_{u ∈ MA_s} \langle u, r \rangle \}$$

of the mapped atomic set $MA_s ⊂ \mathbb{R}^m$. Because the exposed faces decompose under set addition, it follows from the expression (1.5) of $A_s$ that

$$F(MA_s; r) = \sum_{i=1}^{k} F(λ_iMA_i; r).$$

Thus, the facial exposure operation on Line 3 can be computed by separately exposing faces on each of the individual mapped atomic sets, which can be implemented in parallel, i.e.,

$$p^{(t)} = τ \sum_{i=1}^{k} λ_ip_i^{(t)} \quad \text{where} \quad p_i^{(t)} ∈ F(MA_i; r^{(t)}) \quad ∀i ∈ 1:k.$$

The conditional-gradient method converges to the required optimality within $O(1/ε)$ iterations [21]. Combined with the complexity of the level-set method, we thus expect a total worst-case complexity of $O(\log(1/ε)/ε)$ iterations to satisfy the optimality condition (5.2).

5.3. Exposing the signals. Once Algorithm 5.1 reaches Line 7, the residual vector $r^{(t)}$ contains information about the atoms that are in the support of each of the approximations $x_i$ to the signals $x_i^\ast$. It follows from Fan et al. [15, Theorem 7.1] that for all $i ∈ 1:k$,

$$x_i^\ast ∈ \text{cone} F(MA_i; r^\ast), \quad r^\ast := b - M \sum_{i=1}^{k} x_i^\ast.$$
Thus, a solution of the deconvolution problem \( P_2 \) can be recovered by solving

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} \| M \sum_{i=1}^{k} x_i - (b - \bar{r}) \|_2^2 \\
\text{subject to} & \quad x_i \in \text{cone} \mathcal{F}(M A_i; \bar{r}),
\end{align*}
\]

which can be implemented as a standard linear least-squares problem over the coefficients of the atoms exposed in each of the atomic sets.

6. Experiments and novel applications. In section 6.1 we verify empirically the bounds in Theorem 4.4 through a set of numerical experiments on recovering multiple randomly-rotated sparse signals from noisy measurements. Note that the random-rotation requirement in Theorem 4.4 is meant to guarantee the incoherence among the vectors \( \{x_i^k\}_{i=1}^k \). If these signals are already incoherent, the random rotation requirement isn’t required, as verified by other experiments in this section. In section 6.2 we separate a sparse signal and sparse-in-frequency signal. In section 6.3 we separate the superposition of three signals: a sparse signal, a low-rank signal, and noise. In section 6.4 we separate a multiscale low-rank synthetic image.

The algorithm described in section 5 is implemented in the Julia language \([5]\) using version 1.5. All the experiments are conducted on a Linux server with 8 CPUs and 64Gb memory.

6.1. Stability of Demixing. This is experiment is designed to numerically verify the bounds established by Theorem 4.4. The experiment draws multiple realizations of a random problem specified over a range of parameters \( k \) (number of signals), \( m \) (number of measurements), and \( n \) (signal dimension); the sparsity level for each signal is fixed at \( s = 5 \). Each signal \( x^k_i \) in (1.1) is generated according to Assumption 4.2, where each vector \( \hat{x}^k_i \) is \( s \)-sparse.

By construction, the atomic sets \( i \in 1:k \) are defined to be

\[
A_i = Q_i \{ \pm e_1, \ldots, \pm e_n \}
\]

where \( Q_i \sim \text{uniform} \{ \text{SO}(n) \} \).

The upper bound on the statistical dimension of the descent cone for \( (x^k_i, A_i) \), and thus for the descent cone at \( (\hat{x}^k_i, \hat{A}_i) \), for \( s \)-sparse vectors is given by

\[
\phi_1(n, s) := 2s \log \left( \frac{n}{s} \right) + \frac{5}{4}s \geq \delta(D_i);
\]

see \([11]\). The number of signals to demix ranges over \( k \in \{2, 7\} \). In each phase transition plot, the horizontal axis represents the signal dimension \( n \in \{50, 65, \ldots, 500\} \). The vertical axis represents the number of measurements \( m \in \{50, 65, \ldots, 500\} \). The colormap indicates the empirical average over 100 trials of the maximum relative error \( \maxerr / \|\eta\| \), where

\[
\maxerr := \max_{i \in 1:k} \|x^*_i - \hat{x}^k_i\|,
\]

and \( \eta \) is distributed as a standard normal random \( m \)-vector. The red solid curve and the yellow dashed line, respectively, indicate the graphs of the functions

\[
m = k \phi_1(n, s) \quad \text{and} \quad n = \frac{1}{2} k \phi_1(n, s).
\]

We examine the results relative to Theorem 4.4. This result requires \( m > \Delta + 1 \). All points above the solid red curve satisfy the bound \( m > k \phi_1(n, s) \geq \Delta + 1 \). As we can see from
Figure 4. Phase-transition plots for demixing the sum of randomly-rotated sparse signals \( \{x^k_i\}_{i=1}^k \) from noisy measurements \( b \). The top row (left to right) corresponds to \( k = 2, 3, 4 \) respectively, and the bottom row (left to right) corresponds to \( k = 5, 6, 7 \) respectively. The horizontal and vertical axes, respectively, represent the signal dimension \( n \) and measurement dimension \( m \). The colormaps indicate the empirical average of the maximum relative recovery difference (6.2). The red solid curve represents the mapping \( m = k\phi_1(n, s) \) and the yellow dashed line represents the position \( n = k\phi_1(n, s)/2 \), where \( \phi_1 \) is defined by (6.1).

Figure 4, the red solid curves closely match the vertical separations between light and dark regions in the phase-transition plots. Next, consider the yellow dashed line. Theorem 4.4 requires \( n > 4c^2\Delta + 1 \), where \( c \) is some unknown universal constant. We use \( n = \frac{1}{2}k\phi_1(n, s) \) to approximate this bound, shown by the yellow dashed line. As we can see from Figure 4, these dashed lines track the horizontal separation between light and dark regions in the phase transition plots. Theorem 4.4 guarantees that \( \text{maxerr} \) satisfies with high probability the bound (4.2), with \( \alpha = \|\eta\| \). Fix \( (k, s, n) \). The bound decreases as \( m \) increases, which coincides with experimental results. As we can see from Figure 4, the color always goes from dark to light as \( m \) increases. The bound doesn’t change monotonically as \( n \) increases because \( \Delta \) increases as \( n \) increases, as shown in (6.1). This illustrates why the color doesn’t change monotonically as \( n \) increases.

6.2. Separation of sparse signal and sparse-in-frequency signal. We make a direct comparison to the approach described by McCoy et al. [24] and reproduce their experiment on separating an astronomical image into a sparse and a sparse-in-frequency signals. An \( n \)-vector \( x \) is sparse-in-frequency if its discrete cosine transform (DCT) \( Dx \) is sparse, where the orthog-
Figure 5. The star-galaxy separation experiment features two distinct signal components. The image size is $601 \times 601$ pixels.

The star-galaxy image shown in Figure 5 exemplifies this superposition: the stars are well-represented by sparse matrices in $A_s$, and the galaxy component is well-represented by sinusoidal elements in $A_d$. The image size is $601 \times 601$. The results of the separation are shown in the second two panels of Figure 5.

6.3. Sparse and low rank matrix decomposition with structured noise. In this next example we decompose an image that contains a sparse foreground, a low-rank background, and structured noise. This is an example of sparse principle component analysis [16,17,30,37]. Typically, the entry-wise 1-norm and the nuclear norm are used to extract from the matrix each of these qualitatively different structures. Here, we treat the noise as its own signal that also needs to be separated. We consider the observation

$$B = X_s^\sharp + X_l^\flat + X_n^\natural,$$

where $X_s^\sharp \in \mathbb{R}^{m \times n}$ is sparse, $X_l^\flat \in \mathbb{R}^{m \times n}$ is low-rank matrix, and $X_n^\natural \in \mathbb{R}^{m \times n}$ represents structured noise so that $PX_n^\natural Q$ is sparse, where $P$ and $Q$ are random orthogonal $m$-by-$m$ matrices. Based on the atomic framework, we choose the atomic sets for $X_s^\sharp$, $X_l^\flat$, and $X_n^\natural$, respective, as

$$A_s = \{ \pm E_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n \},$$

$$A_l = \{ uv^\top \mid u \in \mathbb{R}^m, v \in \mathbb{R}^n, \| u \| = \| v \| = 1 \},$$

$$A_n = P^\top A_s Q^\top,$$

where $E_{i,j}$ is a $m \times n$ matrix with a single nonzero entry $(i,j)$ with value 1.

For the numerical experiment, we consider the noisy chess board in-painting problem. The chess foreground is sparse and the chess board background is low rank. The image size is $596 \times 596$. The experiment result is shown in Figure 6.
Figure 6. Noisy chess board in-painting experiment. The image size is 596 × 596. Northwest: noisy observation; Northeast: recovered sparse component; Southwest: recovered low rank component; Southeast: denoising result.

Figure 7. Multiscale low rank matrix decomposition experiment. The matrix size is 64 × 64. From left to right: observation; recovered $P_i$-block-wise low rank component for $i = 1, \ldots, 4$. All the blocks in $P_i$ have the same size $4^{i-1} × 4^{i-1}$ for $i = 1, \ldots, 4$.

6.4. Multiscale low rank matrix decomposition. The multiscale low-rank matrix decomposition problem proposed by Ong and Lustig [28] generalizes the sparse and low-rank matrix decomposition through a block-wise low-rank structure. Let $X$ be an $m × n$ matrix and $\mathcal{P}$ be a partition of $X$ into multiple blocks. Then $X$ is considered to be block-wise low-rank with respect to $\mathcal{P}$ if all the blocks are low rank. For each block $p \in \mathcal{P}$ with size $m_p × n_p$, let $X_p$ denote the corresponding part of the matrix $X$ and let $R_p : \mathbb{R}^{m×n} \rightarrow \mathbb{R}^{m_p×n_p}$ denote the linear operator that can extract $X_p$ from $X$, namely $R_p(X) = X_p$. The adjoint operator
$R_p^*: \mathbb{R}^{m_p \times n_p} \rightarrow \mathbb{R}^{m \times n}$ embeds an $m_p \times n_p$ matrix into a $m \times n$ zero matrix. With this operator,

$$X = \sum_{p \in P} R_p^*(X_p).$$

Each block-wise low-rank signal is represented by a corresponding atomic set. By definition, each block $X_p \in \mathbb{R}^{m_p \times n_p}$ is low rank, and thus $X_p$ is $A_p$-sparse, where

$$A_p = \{uv^\top | u \in \mathbb{R}^{m_p}, v \in \mathbb{R}^{n_p}, \|u\| = \|v\| = 1\}.$$

One and Lustig [28] propose a block-wise nuclear norm and its associated dual norm, respectively, by the functions

$$\|\cdot\|_p = \sum_{p \in P} \|R_p(\cdot)\|_1, \quad \|\cdot\|_{p,\infty} = \max_{p \in P} \|R_p(\cdot)\|_\infty,$$

where $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are the Schatten 1- and $\infty$-norms of their matrix arguments. It follows that the block-wise norm $\|\cdot\|_p$ and dual norm $\|\cdot\|_{p,\infty}$ are the gauge and support functions, respectively, for the atomic set $A_P := \bigcup_{p \in P} R_p^* A_p$.

We reproduce the synthetic model described by Ong and Lustig, who construct the superposition $B = \sum_{i=1}^k X_i^n$, where $X_i^n \in \mathbb{R}^{m \times n}$ is block-wise low rank with respect to the multiscale partitions $\{P_i\}_{i=1}^k$. In our experiment, we set $m = n = 64$, $k = 4$, and for each $i \in 1:k$,

$$m_p = n_p = 4^{i-1} \quad \forall p \in P_i.$$

At the lowest scale $i = 1$, a block-wise low-rank matrix is a scalar, and so 1-sparse matrices are included with the atomic set $A_{P_1}$. The solutions of the deconvolution procedure (P2) are shown in Figure 7.

7. Proofs. This section contains proofs for the mathematical statements in section 3 and section 4. We begin with several technical results needed for analysis, which describe useful properties of descent cones. Some of these results contain their own intrinsic interest.

7.1. Lemmas.

Lemma 7.1 (Properties of descent cones). Let $A$, $A_1$, $A_2$ be compact sets in $\mathbb{R}^n$ that contain the origin in their interiors. Fix the vectors $x, x_1, x_2$. The following properties hold.

a) A vector $d$ is contained in $D(A, x)$ if and only if there is some $\bar{\alpha} > 0$ such that $\gamma_{A_\alpha}(x + ad) \leq \gamma_{A}(x)$ for all $\alpha \in [0, \bar{\alpha}]$;

b) $D(\tau A, x) = D(A, x)$ $\forall \tau > 0$;

c) $D(QA, Qx) = QD(A, x)$ if $Q \in SO(n)$;

d) $D(A_1 + A_2, x_1 + x_2) \leq D(A_1, x_1) + D(A_2, x_2)$ if $\gamma_{A_1}(x_1) = \gamma_{A_2}(x_2)$.

Proof.

a) See [26, Proposition 2.5];

b) Because a gauge function is positive homogenous,

$$D(\tau A, x) = \text{cone} \{ d \mid \gamma_{\tau A}(x + d) \leq \gamma_{\tau A}(x) \} = \text{cone} \{ d \mid \gamma_A(x + d) \leq \gamma_A(x) \}.$$
c) Because \( \gamma_{Q,A} = \gamma_A(Q^*) \),

\[
D(Q,A,Qx) = \text{cone}\{ d \mid \gamma_{Q,A}(Qx + d) \leq \gamma_{Q,A}(Qx) \}
\]

\[
= \text{cone}\{ d \mid \gamma_A(x + Q^*d) \leq \gamma_A(x) \}
\]

\[
= \text{cone}\{ Qd' \mid \gamma_A(x + d') \leq \gamma_A(x) \} = QD(A,x).
\]

d) By definition of descent cone and polar convolution,

\[
D(A_1 + A_2, x_1 + x_2)
\]

\[
= \text{cone}\{ d \mid \gamma_{A_1+A_2}(x_1 + x_2 + d) \leq \gamma_{A_1+A_2}(x_1 + x_2) \}
\]

\[
= \text{cone}\left\{ d \mid \inf_{d_1 + d_2 = d} \max \{ \gamma_{A_1}(x_1 + d_1), \gamma_{A_2}(x_2 + d_2) \} \leq \gamma_{A_1+A_2}(x_1 + x_2) \right\}.
\]

For every \( d \in D(A_1 + A_2, x_1 + x_2) \), by Lemma 7.1(a), there exists \( \alpha > 0 \) such that

\[
\gamma_{A_1+A_2}(x_1 + x_2 + ad) \leq \gamma_{A_1+A_2}(x_1 + x_2).
\]

Then there exists \( d_1, d_2 \) such that \( d_1 + d_2 = ad \) and

\[
\max \{ \gamma_{A_1}(x_1 + d_1), \gamma_{A_2}(x_2 + d_2) \} \leq \gamma_{A_1+A_2}(x_1 + x_2) \leq \max \{ \gamma_{A_1}(x_1), \gamma_{A_2}(x_2) \}.
\]

By the assumption \( \gamma_{A_1}(x_1) = \gamma_{A_2}(x_2) \), it follows that \( d_i \in D(A_i, x_i) \), which implies \( ad = d_1 + d_2 \in D(A_1, x_1) + D(A_2, x_2) \). Thus \( d \in D(A_1, x_1) + D(A_2, x_2) \).

The Gaussian width and spherical width of a convex cone are defined by

\[
\omega(K) = \mathbb{E}_g \sup \{ \langle g, y \rangle \mid y \in K \cap S^{n-1} \} \quad \text{and} \quad \omega_s(K) = \mathbb{E}_\theta \sup \{ \langle \theta, y \rangle \mid y \in K \cap S^{n-1} \}.
\]

The expectations in the definition of the Gaussian width and spherical width, respectively, are taken with respect to the standard Gaussian normal \( \text{normal}(0, I_n) \) and with respect to the uniform distribution \( \text{uniform}(S^{n-1}) \). Many of the results that we use rely either on properties of the statistical dimension or the above widths. The following lemma summarizes the main properties that we use regarding the relationship between the conic summaries \( \omega, \omega_s \), and \( \delta \).

**Lemma 7.2 (Properties of conic statistical summaries).** Let \( K, K_1, K_2 \) be some closed and convex cones in \( \mathbb{R}^n \) and let \( Q \in SO(n) \). Then the following properties hold.

a) \( \delta(QK) = \delta(K) \);

b) \( \delta(K) + \delta(K^c) = n \);

c) \( \delta(K_1 \cup K_2) + \delta(K_1 \cap K_2) = \delta(K_1) + \delta(K_2) \);

d) \( \delta(K_1 + K_2) \leq \delta(K_1) + \delta(K_2) \);

e) \( \text{If } K_1 \subseteq K_2 \text{, then } \delta(K_1) \leq \delta(K_2) \);

f) \( \omega^2(K) \leq \delta(K) \leq \omega^2(K) + 1 \);

g) \( (n - 1) \omega_s^2(K) \leq \delta(K) \leq n \omega_s^2(K) + 1 \).

**Proof.** See [2, Proposition 3.1(6), Proposition 3.1(8), Section 5.6(2), Proposition 3.1(10) and Proposition 10.2], respectively, for (a), (b), (c), (e) and (f).
d) It follows from [32, Corollary 16.4.2] that \((K_1 + K_2)^o = K_1^o \cap K_2^o\). Then
\[
\delta(K_1 + K_2) = n - \delta((K_1 + K_2)^o) \\
= n - \delta(K_1^o \cap K_2^o) \\
= n - (\delta(K_1^o) + \delta(K_2^o) - \delta(K_1^o \cup K_2^o)) \\
= n - ((n - \delta(K_1)) + (n - \delta(K_2)) - \delta(K_1^o \cup K_2^o)) \\
= \delta(K_1) + \delta(K_2) - (n - \delta(K_1^o \cup K_2^o)) \leq \delta(K_1) + \delta(K_2),
\]
where the first and fourth equalities follow from Lemma 7.2(b), the third equality follows from Lemma 7.2(c) and the last inequality follows from the fact \(\delta(K_1^o \cup K_2^o) \leq n\).

g) First, we prove that \(\sqrt{n - 1} \omega_s(K) \leq \omega(K) \leq \sqrt{n} \omega_s(K)\). Let \(g \sim \text{normal}(0, I_n)\) and \(\theta \sim \text{uniform}(S^{n-1})\). From the definition of conic Gaussian width,
\[
\omega(K) = \mathbb{E}_g \sup_{y \in K \cap S^{n-1}} \langle g, y \rangle \\
= \mathbb{E}_g \|g\| \sup_{y \in K \cap S^{n-1}} \left\langle \frac{g}{\|g\|}, y \right\rangle \\
= \mathbb{E}_g \|g\| \mathbb{E}_\theta \sup_{y \in K \cap S^{n-1}} \langle \theta, y \rangle = \mathbb{E}_g \|g\| \cdot \omega_s(K),
\]
where the third equality follows from the fact that \(g/\|g\|\) is uniformly distributed on \(S^{n-1}\) and the last equality follows from the definition of spherical width. By [10, Section 3.1],
\[
\sqrt{n - 1} \leq \mathbb{E}_{g \sim \mathcal{N}(0, I_n)} \|g\| \leq \sqrt{n}.
\]
Therefore, \(\sqrt{n - 1} \omega_s(K) \leq \omega(K) \leq \sqrt{n} \omega_s(K)\). Now combine with Lemma 7.2(f) to obtain
\[(n - 1) \omega_s^2(K) \leq \delta(K) \leq n \omega_s^2(K) + 1.\]

The next result establishes the relationship between the statistical dimension of the descent cone to the aggregate signal \(x_s^b\) and the descent cones of the constituent signals \(x_i^b\).

Lemma 2.1. From the definitions of \(x_s^b\) and \(A_s\),
\[
\delta(D(A_s, x_s^b)) = \delta \left( D \left( \sum_{i=1}^k \lambda_i A_i, \sum_{i=1}^k x_i^b \right) \right) \leq \delta \left( \sum_{i=1}^k D(\lambda_i A_i, x_i^b) \right) \leq \sum_{i=1}^k \delta \left( D(A_i, x_i^b) \right),
\]
where the first inequality follows from Assumption 1.1, Lemma 7.1(d) and Lemma 7.2(e); and the second inequality follows from Lemma 7.1(b) and Lemma 7.2(d). 

7.2. Proof for Proposition 3.2. For each \(i \in 1:k\), let \(\epsilon_i := x_i^b - x_i^b\) and \(\epsilon_{-i} := \sum_{j \neq i} \epsilon_j\). By the definition of descent cone, \(\epsilon_i \in D(A_i, x_i^b)\). Because \(\{(x_i^b, A_i)\}_{i=1}^k\) are \(\beta\)-incoherent for some \(\beta \in (0, 1]\), by Definition 3.1,
\[
\cos \angle (-\epsilon_i, \epsilon_{-i}) \leq 1 - \beta.
\]
It follows that
\[ \| e_i + e_{-i} \|^2 = \| e_i \|^2 - 2 \cos \angle (-e_i, e_{-i}) \| e_i \| \| e_{-i} \| + \| e_{-i} \|^2 \]
\[ \geq \| e_i \|^2 - 2(1 - \beta)\| e_i \| \| e_{-i} \| + \| e_{-i} \|^2 \]
\[ = \beta (\| e_i \|^2 + \| e_{-i} \|^2) + (1 - \beta)(\| e_i \| - \| e_{-i} \|)^2 \geq \beta \| e_i \|^2. \]

The desired result follows.

7.3. Proof for Proposition 4.1. Throughout this proof, \( Q \) is a random matrix drawn uniformly from \( \text{SO}(n) \). Define \( \tilde{K}_i := K_i \cap S^{n-1} \) for \( i = 1, 2 \). Fix \( x \in \tilde{K}_1 \). By the definition of spherical width and the fact that \( Qx \sim \text{uniform}(S^{n-1}) \),
\[ \mathbb{E}_Q \sup \{ \langle Qx, y \rangle \mid y \in \tilde{K}_2 \} = \omega_s(K_2). \]

Note that the function
\[ \theta \mapsto \sup \{ \langle \theta, y \rangle \mid y \in \tilde{K}_2 \} \]
is 1-Lipschitz on \( S^{n-1} \). By the concentration inequality of Lipschitz functions on the unit sphere [22, Proposition 1.7],
\[ \mathbb{P}_Q \left[ \sup \{ \langle Qx, y \rangle \mid y \in \tilde{K}_2 \} \geq \omega_s(K_2) + t \right] \leq \exp(-\frac{1}{2}(n - 1)t^2). \]

Combine with Lemma 7.2(g) to deduce that
\[ (7.1) \quad \mathbb{P}_Q \left[ \sup \{ \langle Qx, y \rangle \mid y \in \tilde{K}_2 \} \geq \sqrt{\frac{\delta(K_2)}{n - 1} + t} \right] \leq \exp(-\frac{1}{2}(n - 1)t^2). \]

Next, choose an \( \epsilon \)-net \( \mathcal{N}_\epsilon \) of \( \tilde{K}_1 \). By [10, Theorem 3.5],
\[ \omega(K_1) \geq c\epsilon \sqrt{\log(|\mathcal{N}_\epsilon|)}, \]
where \( c > 0 \) is a universal constant. Note that the similar bound can also be obtained from [41, Corollary 7.4.3]. Now combine with Lemma 7.2(f) to conclude that
\[ (7.2) \quad |\mathcal{N}_\epsilon| \leq \exp \left( \frac{\delta(K_1)}{c^2\epsilon^2} \right). \]

Next, it follows from (7.1) and (7.2) that
\[ \mathbb{P}_Q \left[ \sup \{ \langle Qx, y \rangle \mid x \in \mathcal{N}_\epsilon, \ y \in \tilde{K}_2 \} \geq \sqrt{\frac{\delta(K_2)}{n - 1} + t} \right] \leq \exp \left( \frac{\delta(K_1)}{c^2\epsilon^2} \right) \exp(-\frac{1}{2}(n - 1)t^2). \]

Because \( \mathcal{N}_\epsilon \) is an \( \epsilon \)-net of \( \tilde{K}_1 \),
\[ \sup \{ \langle Qx, y \rangle \mid x \in \tilde{K}_1, \ y \in \tilde{K}_2 \} - \epsilon \leq \sup \{ \langle Qx, y \rangle \mid x \in \mathcal{N}_\epsilon, \ y \in K_2 \cap S^{n-1} \}. \]
It follows that

\[
\mathbb{P}_Q \left[ \sup_{x \in K_1^*, y \in K_2^*} \langle Qx, y \rangle \geq \sqrt{\frac{\delta(K_2)}{n-1}} + \epsilon + t \right] \leq \exp \left( \frac{\delta(K_1)}{c^2 \epsilon^2} \right) \exp \left( -\frac{1}{2} (n-1)t^2 \right).
\]

Finally, let \( t = \frac{1}{2} \) and \( \epsilon = \frac{4}{c} \sqrt{\frac{\delta(K_1)}{n-1}} \). Then

\[
\sqrt{\frac{\delta(K_2)}{n-1}} + \epsilon + t \leq 2 \max \left\{ 1, \frac{4}{c} \right\} \sqrt{\frac{\delta(K_1) + \delta(K_2)}{n-1}} + \frac{1}{2}
\]

and

\[
\exp \left( \frac{\delta(K_1)}{c^2 \epsilon^2} \right) \exp \left( -\frac{1}{2} (n-1)t^2 \right) = \exp \left( -\frac{1}{16} (n-1) \right).
\]

Combine these last two displayed equations with (7.3) and let \( \hat{c} = 2 \max \left\{ 1, \frac{4}{c} \right\} \), to deduce

\[
\mathbb{P}_Q \left[ \cos \angle(QK_1, K_2) \geq \hat{c} \sqrt{\frac{\delta(K_1) + \delta(K_2)}{n-1}} + \frac{1}{2} \right] \leq \exp \left( -\frac{1}{16} (n-1) \right),
\]

which completes the proof.

### 7.4. Proof for Proposition 4.3.

Throughout this proof, define \( \hat{D}_i := D(\hat{A}_i, \hat{x}_i^Q) \) and \( \delta_i = \delta(D_i) \) for all \( i \in 1:k \). Firstly, by Lemma 7.1(c),

\[
D_i = D(Q_i \hat{A}_i, Q_i \hat{x}_i^Q) = Q_i \hat{D}_i, \quad \forall i = 1, \ldots, k.
\]

Combining with Lemma 7.2(a), it follows that

\[
\delta(\hat{D}_i) = \delta(Q_i^T D_i) = \delta_i, \quad \forall i = 1, \ldots, k.
\]

Next, for each \( i \in 1:k \),

\[
\cos \angle \left( -D_i, \sum_{j \neq i} D_j \right) = \cos \angle \left( -Q_i \hat{D}_i, \sum_{j \neq i} Q_j \hat{D}_j \right).
\]

By Lemma 7.2(a) and Lemma 7.2(d),

\[
\delta(-\hat{D}_i) + \delta \left( \sum_{j \neq i} Q_j \hat{D}_j \right) \leq \delta(\hat{D}_i) + \sum_{j \neq i} \delta(Q_j \hat{D}_j) \leq \sum_{i=1}^k \delta_i = \Delta.
\]

Next, for each \( q \in 1:k \), define the event

\[
E_q = \left\{ \cos \angle \left( -Q_q \hat{D}_q, \sum_{j \neq q} Q_j \hat{D}_j \right) \geq 1 - \beta \right\},
\]
where $\beta$ is defined in (4.1). Denote the indicator random variable for $E_q$ by $\mathbb{1}_{E_q}$, which evaluates to 1 if $E_q$ occurs and otherwise evaluates to 0. Let $\{Q_i\}_{i=1}^k$ be a set of random matrices drawn uniformly from $\text{SO}(n)$. Then, the following chain of inequalities gives the upper bound for the probability of the event $E_q$:

$$
P\{Q_i\}_{i=1}^k (E_q) = \mathbb{E}_{\{Q_i\}_{i=1}^k} \mathbb{1}_{E_q}$$

$$\leq \mathbb{E}_{\{Q_i\}_{i=1}^k} \mathbb{E}_{Q_q} \left( \mathbb{1}_{E_q} \mid Q_j \forall j \neq q \right)$$

$$\leq \mathbb{E}_{\{Q_i\}_{i=1}^k} \mathbb{E}_{Q_q} \left[ \cos \angle \left( -Q_q \hat{D}_q, \sum_{j \neq q} Q_j \hat{D}_j \right) \right] \geq 1 - \beta$$

$$\leq \mathbb{E}_{\{Q_i\}_{i=1}^k} \exp \left( -\frac{1}{16} (n-1) \right) = \exp \left( -\frac{1}{16} (n-1) \right).$$

Equality (a) follows from the definition of an indicator random variable. Equality (b) follows from the fact that the random matrices $\{Q_i\}_{i=1}^k$ are independent and from properties of conditional expectation. Equality (c) follows from the property of the expectation of the indicator random variable $\mathbb{1}_{E_q}$ with $\{Q_j\}_{j \neq q}$ fixed. Finally, inequality (d) follows from (7.4) and Proposition 4.1.

Extend the bound on $P\{Q_i\}_{i=1}^k (E_q)$ to all $q \in 1:k$ via the union bound:

$$P\{Q_i\}_{i=1}^k (E_1 \land \cdots \land E_k) \leq k \exp \left( -\frac{1}{16} (n-1) \right).$$

Rewrite this inequality to obtain

$$P\{Q_i\}_{i=1}^k \left[ \cos \angle \left( -Q_q \hat{D}_q, \sum_{j \neq q} Q_j \hat{D}_j \right) \leq 1 - \beta \ \forall q \in 1:k \right] \geq 1 - k \exp \left( -\frac{1}{16} (n-1) \right).$$

Finally, the assumption $\Delta < (n-1)/4c^2$ implies that $\beta \in (0, 1)$. With Definition 3.1, we can thus assert that $\{(x_i^k, A_i)\}_{i=1}^k$ are $\beta$-incoherent with probability at least $1 - k \exp \left( -\frac{1}{16} (n-1) \right)$.

**8. Looking ahead.** The random rotation model is a useful mechanism for introducing incoherence among the individual signals. However, even in contexts where it’s possible to rotate the signals, it may prove too costly to do so in practice because the rotations need to be applied at each iteration of the algorithm in Algorithm 5.1. We might then consider other mechanisms for introducing incoherence that are computationally cheaper, and rely instead, for example, on some fast random transform. The literature on demixing abounds with various incoherence notions. We wish to explore what is the relationship between these and definition of $\beta$-incoherence that we adopt. Alternative incoherence definitions may prove useful in deriving other mechanisms for inducing incoherence in the signals.

A significant assumption of our analysis is that the parameters $\lambda_i$ exactly equilibrate the gauge values for each signal; cf. (1.6). In practice, however, we can only estimate these. It may be possible to analyze how the stability in the recovery of the signals depends on errors that might exist in the ideal parameter choices.
REFERENCES


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