Algorithms for sparse optimization

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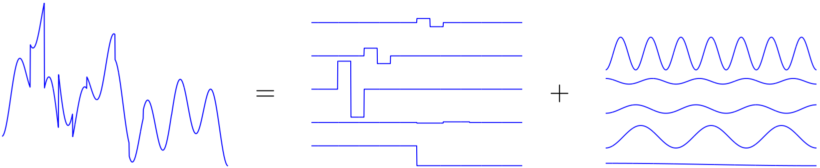
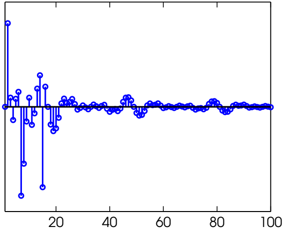
Sparse representations

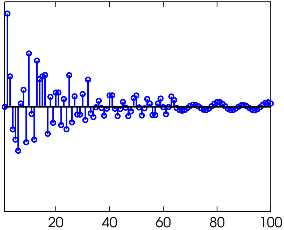


$y = Hx$



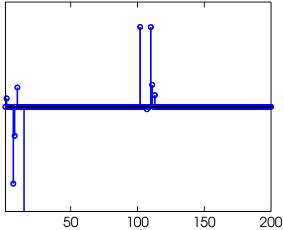
Haar

$y = Dx$



DCT

$y = \begin{bmatrix} H & D \end{bmatrix} x$



Haar/DCT dictionary

Sparsity via a transform

Represent a signal y as a superposition of elementary signal atoms:

$$y = \sum_j \phi_j x_j \qquad \text{ie,} \qquad y = \Phi x$$

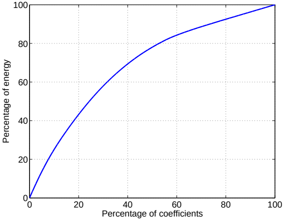
Orthogonal bases yield a unique representation:

$$x = \Phi^T y$$

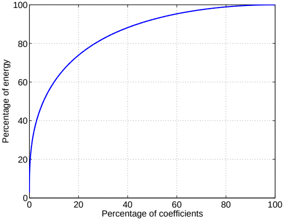
Overcomplete dictionaries, eg, $A = [\Phi_1 \ \Phi_2]$ have more flexibility.
But representation is not unique. One approach:

$$\text{minimize} \quad \text{nnz}(x) \quad \text{subj to} \quad Ax \approx y$$

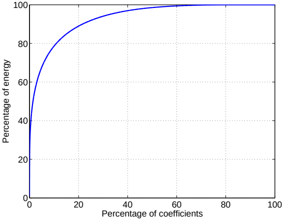
Efficient transforms



Identity

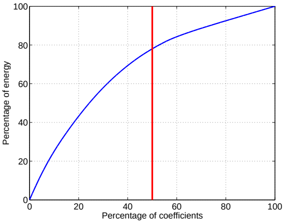


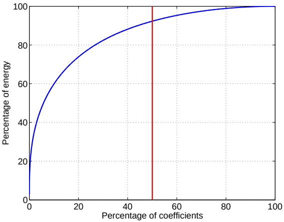
Fourier

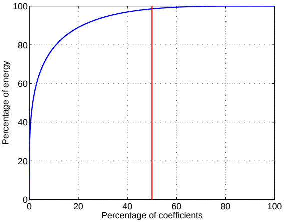


Wavelet

Efficient transforms







Identity

Fourier

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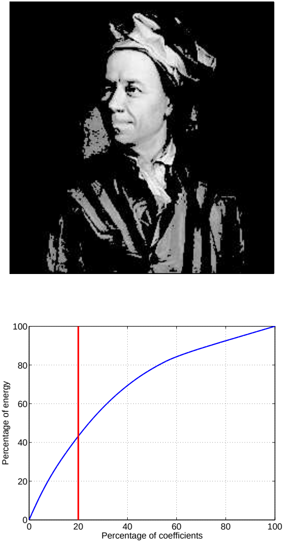
Efficient transforms



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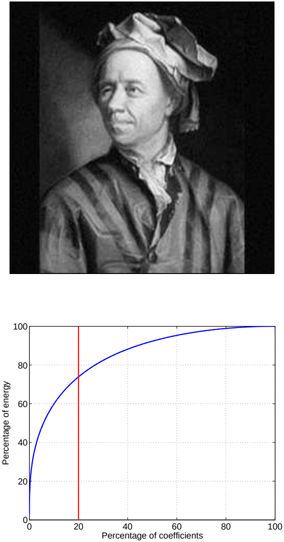
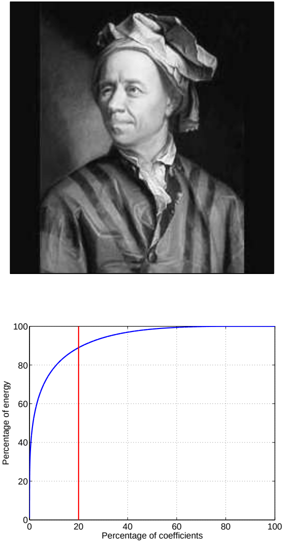


Fourier



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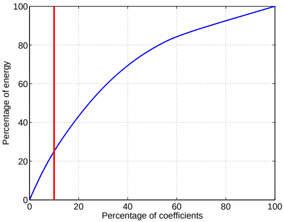
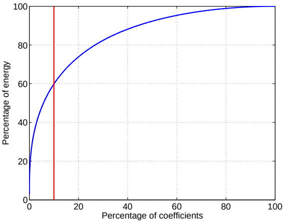
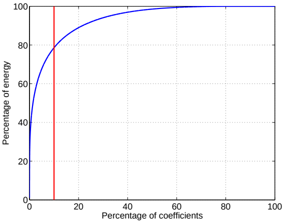
Efficient transforms







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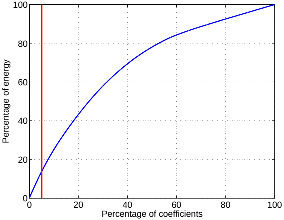

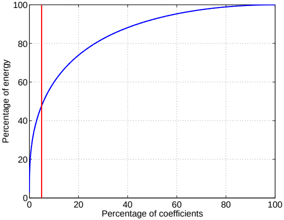

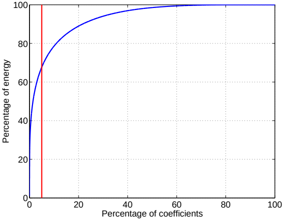

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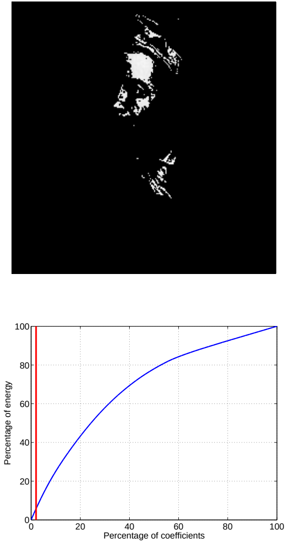
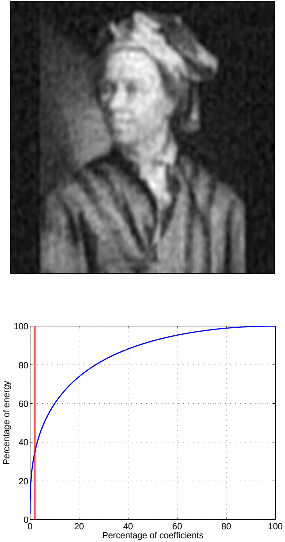
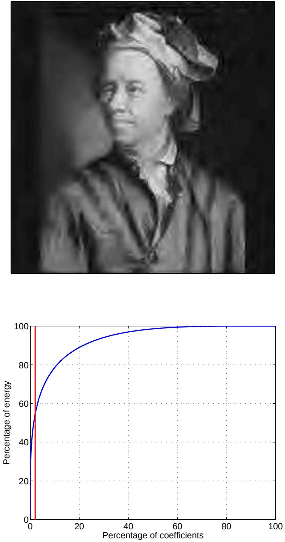


Identity



Fourier



Wavelet

Prototypical workflow

Measure structured signal y via linear measurements:

$$b_i = \langle f_i, y \rangle, \quad i = 1, \ldots, m$$

$$\text{ie,} \quad b = My, \quad M = \boxed{}$$

Decode the m measurements:

$$M\Phi x \approx b, \qquad x \text{ "sparse"}$$

Reconstruct the signal:

$$\widehat{y} := \Phi x$$

Guarantees: number of samples m needed to reconstruct signal w.h.p.

- Compressed sensing (eg, M Gaussian/Φ orthogonal): $\mathcal{O}(k \log \frac{n}{k})$
- matrix completion: # matrix samples $\mathcal{O}(n^{5/4} r \log n)$

Sparse optimization

Problem: find sparse solution x

$$\text{minimize} \quad \text{nnz}(x) \quad \text{subj to} \quad Ax \approx b$$

Convex relaxations, eg:

basis pursuit [Chen et al. (1998); Candès et al. (2006)]

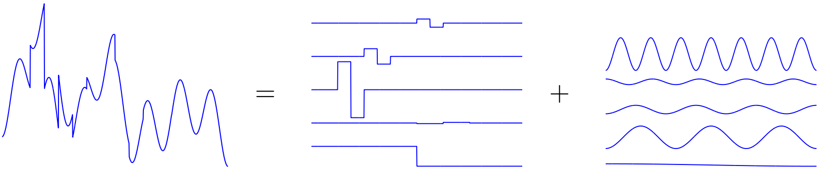
$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad \|x\|_1 \qquad \text{subj to} \quad Ax \approx b \qquad (m \ll n)$$

nuclear norm [Fazel (2002); Candes and Recht (2009)]

$$\underset{X \in \mathbb{R}^{n \times p}}{\text{minimize}} \quad \textstyle\sum_i \sigma_i(X) \quad \text{subj to} \quad \mathcal{A}X \approx b \qquad (m \ll np)$$

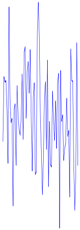
APPLICATIONS and FORMULATIONS

Compressed sensing



$\langle \phi_1, y\rangle, \ldots, \langle \phi_m, y\rangle$

$\boldsymbol{\Phi} y =$



$\boldsymbol{\Phi} = \boxed{\text{Gaussian}}$

$\boldsymbol{\Phi} y \approx \boldsymbol{\Phi} \boxed{\text{Haar}}\boxed{\text{DCT}}\, x \quad \Rightarrow \quad b \approx \boxed{A}\, x$

$$\text{minimize} \quad \|x\|_1 \quad \text{subj to} \quad Ax \approx b$$

Image deblurring

Observe $b = My$, y = image, M = blurring operator

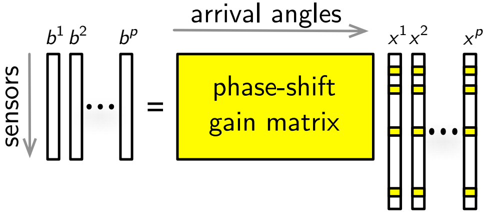
Recover significant coeff's via

$$\text{minimize} \quad \tfrac{1}{2}\|MWx - b\|^2 + \lambda\|x\|_1$$

λ	2.3e+0	2.3e-2	2.3e-4
$\|Ax - b\|$	2.6e+2	3.0e+0	3.1e-2
nnz(x)	72	1,741	28,484

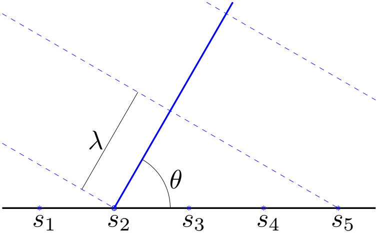
Sparco problem `blurrycam` [Figueiredo et al. (2007)]

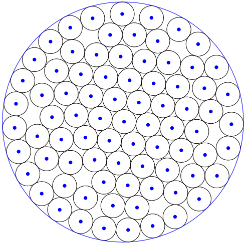
Joint sparsity: source localization

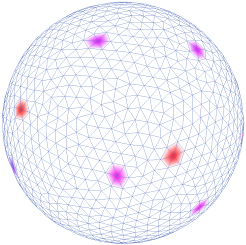


[Malioutov et al. (2005)]

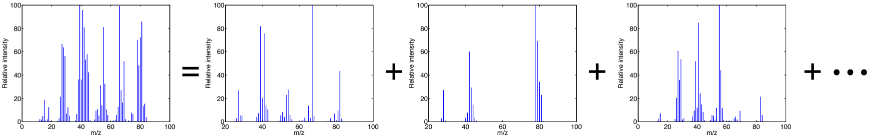
$$\begin{array}{ll} \text{minimize} & \|X\|_{1,2} \\ \text{subj to} & \|B - AX\|_F \le \sigma \end{array}$$



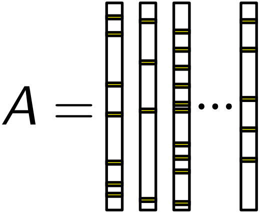




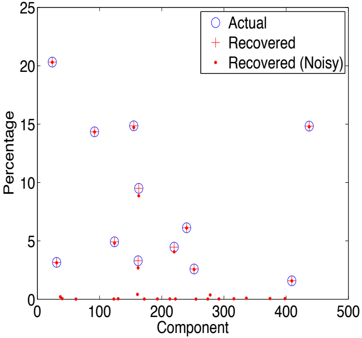
Mass spectrometry – separating mixtures



mixture　　molecule 1　　molecule 2　　molecule 3



spectral fingerprints



minimize $\sum_i x_i$ subj to $Ax \approx b,\ x \geq 0$ [Du-Angeletti '06, Berg-F. '11]

Phase retrieval

Observe signal $x \in \mathbb{C}^n$ through **magnitudes**.

$$b_k = |\langle a_k, x\rangle|^2, \qquad k = 1, \ldots, m$$

Lift.

$$b_k = |\langle a_k, x\rangle|^2 = \langle a_k a_k^*, xx^*\rangle = \langle a_k a_k^*, X\rangle \quad \text{with} \quad X = xx^*$$

Decode. Find X such that

$$\langle a_k a_k^*, X\rangle \approx b_k, \quad X \succeq 0, \quad \operatorname{rank}(X) = 1$$

Convex relaxation.

$$\underset{X \in \mathcal{H}^n}{\text{minimize}} \quad \operatorname{tr}(X) \quad \text{subj to} \quad \langle a_k a_k^*, X\rangle \approx b_k, \quad X \succeq 0$$

[Chai et al. (2011); Candès et al. (2012); Waldspurger et al. (2015)]

Matrix completion

Observe random matrix entries of an $m \times p$ matrix Y.

$$b_k = Y_{i_k,j_k} \equiv e_{i_k}^T Y e_{j_k} = \langle e_{i_k} e_{j_k}^T, Y \rangle, \quad k = 1, \ldots, m$$

Decode. Find $m \times p$ matrix X such that

$$\langle e_{i_k} e_{j_k}^T, X \rangle \approx b_k, \quad \text{rank}(X) = \min$$

Convex relaxation.

$$\underset{X \in \mathbb{R}^{m \times p}}{\text{minimize}} \quad \sum_i \sigma_i(X) \quad \text{subj to} \quad \langle e_{i_k} e_{j_k}^T, X \rangle \approx b_k$$

[Fazel (2002); Candes and Recht (2009)]

CONVEX OPTIMIZATION

Convex optimization

$$\underset{x \in \mathcal{C}}{\text{minimize}} \quad f(x)$$

$f : \mathbb{R}^n \to \mathbb{R}$ is a convex function, $\qquad \mathcal{C} \subseteq \mathbb{R}^n$ is a convex set

Essential objective.

$$f(x) : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\} = \begin{cases} f(x) & \text{if } x \in \mathcal{C} \\ +\infty & \text{otherwise} \end{cases}$$

No loss of generality in restricting ourselves to unconstrained problems:

$$\underset{x}{\text{minimize}} \quad f(x) + g(Ax)$$

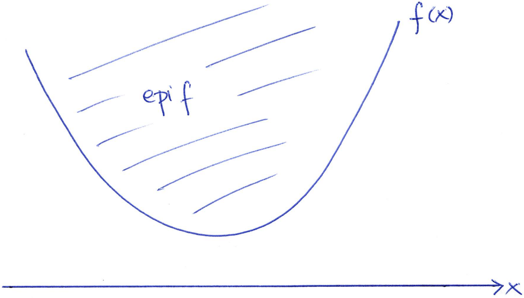
$$f : \mathbb{R}^n \to \overline{\mathbb{R}}, \quad g : \mathbb{R}^n \to \overline{\mathbb{R}}, \quad A \text{ is } m \times n \text{ matrix}$$

Convex functions

Definition. $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is convex if its **epigraph**

$$\operatorname{epi} f = \{ (x, \tau) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \le \tau \}$$

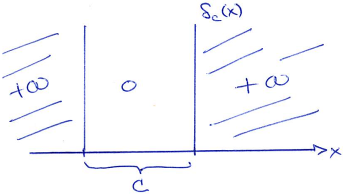
is convex.



Examples

Indicator on a convex set.

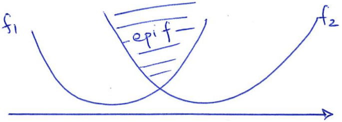
$$\delta_{\mathcal{C}}(x) = \begin{cases} 0 & \text{if } x \in \mathcal{C}, \\ +\infty & \text{otherwise,} \end{cases} \qquad \operatorname{epi} \delta_{\mathcal{C}} = \mathcal{C} \times \mathbb{R}_+$$



Supremum of convex functions.

$$f := \sup_{j \in \mathcal{J}} f_j, \qquad \{f_j\}_{j \in \mathcal{J}} \text{ convex functions}$$

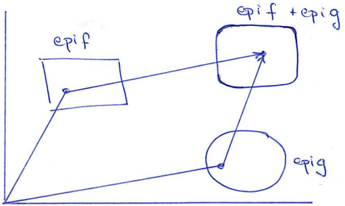
$$\operatorname{epi} f = \bigcap_{j \in \mathcal{J}} \operatorname{epi} f_j$$



Infimal convolution (epi-addition).

$$(f \,\square\, g)(x) = \inf_z \left\{ f(z) + g(x - z) \right\}$$

$$\operatorname{epi}(f \,\square\, g) = \operatorname{epi} f + \operatorname{epi} g$$



Examples

Generic form.

$$\underset{x}{\text{minimize}} \quad f(x) + g(Ax)$$

Basis pursuit.

$$\min_x \quad \|x\|_1 \quad \text{st} \quad Ax = b, \qquad\qquad f = \lambda\|\cdot\|_1, \qquad g = \delta_{\{b\}}$$

Basis pursuit denoising.

$$\min_x \quad \|x\|_1 \quad \text{st} \quad \|Ax - b\|_2 \le \sigma, \qquad f = \|\cdot\|_1, \qquad g = \delta_{\{\|\cdot - b\|_2 \le \sigma\}}$$

Lasso.

$$\min_x \quad \tfrac{1}{2}\|Ax - b\|^2 \quad \text{st} \quad \|x\|_1 \le \tau, \qquad f = \delta_{\tau\mathbb{B}_1}, \qquad g = \tfrac{1}{2}\|\cdot - b\|^2$$

DUALITY

Duality

$$\underset{x}{\text{minimize}} \quad f(x) + g(Ax)$$

Decouple the objective terms:

$$\underset{x,z}{\text{minimize}} \quad f(x) + g(z) \quad \text{subj to} \quad Ax = z$$

Lagrangian function.

$$L(x, z, y) := f(x) + g(z) + \langle y, Ax - z \rangle$$

encapsulates all the information about the problem:

$$\sup_y L(x, z, y) = \begin{cases} f(x) + g(z) & \text{if } Ax = z \\ +\infty & \text{otherwise} \end{cases}$$

$$p^* = \inf_{x,z} \sup_y L(x, z, y)$$

Primal-dual pair.

$$
\begin{aligned}
p^* &= \inf_{x,z} \sup_{y} \; L(x,z,y) \qquad &\text{(primal problem)} \\
d^* &= \sup_{y} \inf_{x,z} \; L(x,z,y) \qquad &\text{(dual problem)}
\end{aligned}
$$

The following **always** holds:

$$p^* \geq d^* \qquad \text{(weak duality)}$$

Under some **constraint qualification**,

$$p^* = d^* \qquad \text{(strong duality)}$$

Fenchel dual

Dual problem.

$$\sup_y \inf_{x,z} \; L(x,z,y) = \sup_y \inf_{x,z} \; f(x) + g(z) + \langle y, Ax - z\rangle$$
$$= \sup_y \left\{ -\sup_x \left[\langle A^T y, x\rangle - f(x)\right] - \sup_z \left[\langle -y, z\rangle - g(z)\right]\right\}$$

Conjugate function.

$$h^*(u) = \sup_x \{\langle u, x\rangle - h(u)\}$$

Fenchel-dual pair.

$$\underset{x}{\text{minimize}} \quad f(x) + g(Ax)$$
$$\underset{y}{\text{minimize}} \quad f^*(A^T y) + g^*(-y)$$

PROXIMAL METHODS

Proximal algorithm

$$\underset{x}{\text{minimize}} \quad f(x) \qquad (g \equiv 0)$$

Proximal operator.

$$\mathbf{prox}_{\alpha f}(x) := \arg\min_{z} \left\{ f(z) + \tfrac{1}{2\alpha}\|z - x\|^2 \right\} \quad \equiv \quad f \,\square\, \tfrac{1}{2\alpha}\|\cdot\|^2$$

Optimality. Every optimal solution x^* solves

$$x = \mathrm{prox}_{\alpha f}(x), \quad \forall \alpha > 0$$

Proximal iteration.

$$x^+ = \mathrm{prox}_{\alpha f}(x)$$

[Martinet (1970)]

$$x_{k+1} = \operatorname{prox}_{\alpha f}(x_k) := \arg\min_z \left\{ f(x) + \frac{1}{2\alpha}\|z - x\|^2 \right\}$$

Descent. Because x_{k+1} is the minimizing argument

$$f(x_{k+1}) + \frac{1}{2\alpha_k}\|x_{k+1} - x_k\|^2 \le f(x) + \frac{1}{2\alpha_k}\|x - x_k\|^2 \quad \forall x$$

Set $x = x_k$:

$$f(x_{k+1}) - f(x_k) \le -\frac{1}{2\alpha_k}\|x_{k+1} - x_k\|^2$$

Convergence.

- finite convergence for f polyhedral [Polyak and Tretyakov (1973)]
- subproblems may be solved approximately [Rockafellar (1976)]
- $f(x_k) - p^* \le \mathcal{O}(1/k)$ [Güler (1991)]
- $f(x_k) - p^* \le \mathcal{O}(1/k^2)$ via an accelerated variant [Güler (1992)]

Moreau-Yosida regularization

M-Y envelope $\quad f_\alpha(x) = \min_z \; f(z) + \frac{1}{2\alpha}\|z-x\|^2$

proximal map $\quad \mathrm{prox}_{\alpha f}(x) = \arg\min_z \; f(z) + \frac{1}{2\alpha}\|z-x\|^2$

Useful properties:

- differentiable: $\quad \nabla f_\alpha(x) = \frac{1}{\alpha}\left[x - \mathrm{prox}_{\alpha f}(x)\right]$
- preserves minima: $\quad \mathrm{argmin}\; f_\alpha = \mathrm{argmin}\; f$
- decomposition: $\quad x = \mathrm{prox}_f(x) + \mathrm{prox}_{f^*}(x)$

Examples

vector 1-norm $\quad \mathrm{prox}_{\alpha\|\cdot\|_1}(x) = \mathrm{sgn}(x) \;.*\; \max\left\{|x| - \alpha, 0\right\}$

Schatten 1-norm $\quad \mathrm{prox}_{\alpha\|\cdot\|_1}(X) = U\bar{\Sigma}V^T, \qquad \bar{\Sigma} = \mathrm{Diag}\left[\mathrm{prox}_{\alpha\|\cdot\|_1}\left(\sigma(X)\right)\right]$

Augmented Lagrangian (AL) algorithm

Rockafellar ('73) connects AL to proximal method on dual

$$\boxed{\text{minimize } f(x) \quad \text{subj to} \quad Ax = b} \qquad \text{ie,} \qquad f(x) + \delta_{\{b\}}(Ax)$$

$$\text{Lagrangian:} \quad L\ (x, y) = f(x) + y^T(Ax - b)$$

$$\text{aug Lagrangian:} \quad L_\rho(x, y) = f(x) + y^T(Ax - b) + \tfrac{1}{2}\rho\|Ax - b\|^2$$

AL algorithm:

$$x^{k+1} = \arg\min_x \ L_\rho(x, y^k) \qquad \text{[may be inexact]}$$

$$y^{k+1} = y^k + \rho(Ax^{k+1} - b) \qquad \text{[multiplier update]}$$

- for linear constraints, AL algorithm $\equiv$ Bregman
- used by `L1-Bregman` for $f(x) = \|x\|_1$ [Yin et al. (2008)]
- proposed by Hestenes/Powell ('69) for smooth nonlinear optimization

SPLITTING METHODS

projected gradient, proximal-gradient, iterative soft-thresholding, alternating projections

Proximal gradient

$$\underset{x}{\text{minimize}} \quad f(x) + g(x)$$

Algorithm.

$$x^+ = \text{prox}_{\alpha g}\left(x - \alpha \nabla f(x)\right)$$

with

$$\alpha \in (\mu, 2/L), \quad \mu > 0, \quad \|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|$$

Convergence.

- $f(x_k) - p^* \leq \mathcal{O}(1/k\ )$ constant stepsize ($\alpha \equiv 1/L$)
- $f(x_k) - p^* \leq \mathcal{O}(1/k^2)$ accelerated variant [Beck and Teboulle (2009)]
- $f(x_k) - p^* \leq \mathcal{O}(\gamma^k)$, $\gamma < 1$, with stronger assumptions

Interpretations

Majorization minimization. Lipshitz assumption on ∇f implies

$$f(z) \le f(x) + \langle \nabla f(x), z - x \rangle + \frac{L}{2}\|z - x\|^2 \quad \forall x, z$$

Proximal-gradient iteration minimizes the majorant: $\forall \alpha \in (0, 1/L)$

$$\begin{aligned} x^+ &= \mathrm{prox}_{\alpha g}\big(x - \alpha \nabla f(x)\big) \\ &\equiv \arg\min_z \Big\{ g(z) + f(x) + \langle \nabla f(x), z - x \rangle + \frac{1}{2\alpha}\|z - x\|^2 \Big\} \end{aligned}$$

Forward-backward splitting.

$$x^+ = \mathrm{prox}_{\alpha g}(x - \alpha \nabla f(x)) \;=\; \underbrace{(I + \alpha \partial g)^{-1}}_{\substack{\text{forward} \\ \text{step}}} \; \underbrace{(I - \alpha \nabla f)}_{\substack{\text{backward} \\ \text{step}}}(x)$$

Projected gradient

$\boxed{\text{minimize } f(x) \text{ subj to } x \in \mathcal{C}}$ $$x^{k+1} = \mathrm{proj}_{\mathcal{C}}(x^k - \alpha^k \nabla f(x^k))$$

LASSO

$$\underset{\|x\|_1 \le \tau}{\text{minimize}} \quad \tfrac{1}{2}\|Ax - b\|^2 \quad \text{via} \quad f(x) = \tfrac{1}{2}\|Ax - b\|^2, \quad g(x) = \delta_{\tau\mathbb{B}_1}(x)$$

- $\mathrm{proj}_{\mathcal{C}}(\cdot)$ can be computed in $\mathcal{O}(n)$ (randomized) [Duchi et al. (2008)]
- used by SPGL1 for Lasso subproblems [vdBerg and Friedlander (2008)]

BPDN (Lagrangian form)

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \ \tfrac{1}{2}\|Ax - b\|^2 + \lambda\|x\|_1 \quad \text{via} \quad \underset{\bar{x} \in \mathbb{R}^{2n}_+}{\text{minimize}} \ \tfrac{1}{2}\|\bar{A}\bar{x} - b\|^2 + \lambda e^T\bar{x}$$

- $\mathrm{proj}_{\ge 0}(\cdot)$ can be computed in $\mathcal{O}(n)$
- GPSR uses Barzilai-Borwein for step α^k [Figueiredo-Nowak-Wright 07]

Iterative soft-thresholding (IST)

$$\boxed{\underset{x}{\text{minimize}} \;\; \tfrac{1}{2}\|Ax - b\|^2 \;+ \lambda\|x\|_1}$$

$$\begin{aligned} x^+ &= \text{prox}_{\alpha\lambda\|\cdot\|_1}(x - \alpha A^T r), \qquad r \equiv Ax - b \\ &= \mathcal{S}_{\alpha\lambda}(x - \alpha A^T r) \end{aligned}$$

where

$$\alpha \in (0,\, 2/\|A\|^2) \quad \text{and} \quad \mathcal{S}_\tau(x) = \text{sgn}(x) \cdot \max\{|x| - \tau,\, 0\}$$

[aka, iterative shrinkage, iterative Landweber]

Derived from various viewpoints:

- expectation-maximization [Figueiredo & Nowak '03]
- surrogate functionals [Daubechies et al '04]
- forward-backward splitting: [Combettes & Wajs '05]
- FISTA (accelerated variant) [Beck-Teboulle '09]

Low-rank approximation via nuclear-norm:

$$\boxed{\text{minimize} \quad \tfrac{1}{2}\|\mathcal{A}X - b\|^2 + \lambda\|X\|_n \quad \text{with} \quad \|X\|_n = \sum \sigma_j(X)}$$

Singular-value soft-thresholding algorithm:

$$x^+ = \mathcal{S}_{\alpha\lambda}(X - \alpha\mathcal{A}^*(r))$$

with

$$r := \mathcal{A}X - b, \qquad \mathcal{S}_\lambda(Z) = U\,\mathrm{diag}(\bar\sigma)V^T, \qquad \bar\sigma = \mathcal{S}_\lambda(\sigma(Z))$$

Computing $\mathcal{S}_\lambda$:

No need to compute full SVD of Z, only leading singular values/vectors

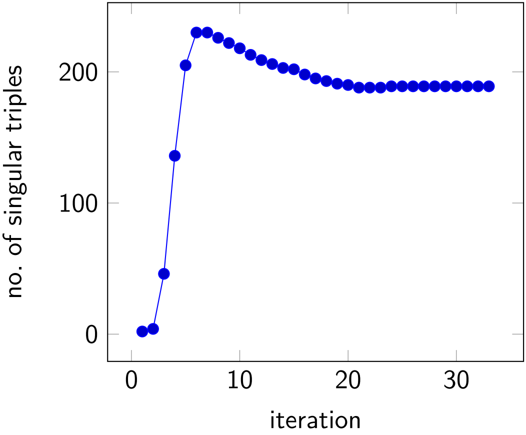
- Monte-Carlo approach [Ma-Goldfarb-Chen '08]
- Lanczos bidiagonalization (via PROPACK) [Cai-Candes-Shen '08; Jain et al '10]

Typical behavior

$$\underset{X \in \mathbb{R}^{n_1 \times n_2}}{\text{minimize}} \quad \|X\|_1 \quad \text{subj to} \quad X_\Omega \approx b$$

recover 1000×1000 matrix X

$\text{rank}(X^*) = 4$



Backward-backward splitting

Approximate one of the functions via its (smooth) Moreau envelope.

$$\underset{x}{\text{minimize}} \quad f_\alpha(x) + g(x) \qquad \text{(both nonsmooth)}$$

$$f_\alpha = f \,\square\, \tfrac{1}{2\alpha}\|\cdot\|^2, \qquad \nabla f_\alpha(x) = \tfrac{1}{\alpha}\left[x - \mathrm{prox}_{\alpha f}(x)\right]$$

Proximal gradient.

$$x^+ = \mathrm{prox}_{\alpha g}\big(x - \alpha \nabla f_\alpha(x)\big) = \mathrm{prox}_{\alpha g}\ \mathrm{prox}_{\alpha f}(x)$$

Projection onto convex sets (POCS).

$$f = \delta_{\mathcal{C}}, \qquad g = \delta_{\mathcal{D}}$$

$$x^+ = \mathrm{proj}_{\mathcal{D}}\ \mathrm{proj}_{\mathcal{C}}(x)$$

solves the problem

$$\underset{x,z}{\text{minimize}} \quad \tfrac{1}{2}\|z - x\|^2 \quad \text{subj to} \quad x \in \mathcal{D},\ z \in \mathcal{C}$$

Alternating direction of method of multipliers (ADMM)

$$\underset{x}{\text{minimize}} \quad f(x) + g(x)$$

ADMM algorithm.

$$\begin{aligned} x^+ &= \text{prox}_{\alpha f}(z - u) \\ z^+ &= \text{prox}_{\alpha g}(x^+ + u) \\ u^+ &= u + x^+ - z^+ \end{aligned}$$

- used by `YALL1` [Zhang et al. (2011)]
- ADMM algo $\equiv$ Douglas-Rachford $\underset{\left(\substack{\text{linear}\\ \text{constraints}}\right)}{\equiv}$ split Bregman
- Esser's UCLA PhD thesis ('10) ; survey by Boyd et al. ('11)

PROXIMAL SMOOTHING

Primal smoothing

$$\underset{x}{\text{minimize}} \quad f(x) + g(Ax)$$

Partial smoothing via the Moreau envelope:

$$\underset{x}{\text{minimize}} \quad f_\mu(x) + g(Ax), \qquad f_\mu = f \,\square\, \tfrac{1}{2\mu}\|\cdot\|^2$$

Example (Basis Pursuit). Huber approximation to 1-norm:

$$f = \|\cdot\|_1, \quad g = \delta_{\{b\}} \quad \Longrightarrow \quad \underset{x}{\text{minimize}} \quad \|x\|_1 \quad \text{subj to} \quad Ax = b$$

Smoothed function is Huber with parameter μ:

$$f_\mu(x) = \begin{cases} x^2/2\mu & \text{if } |x| \le \mu \\ |x| - \mu/2 & \text{otherwise} \end{cases}$$

$$\mu \searrow 0 \quad \Longrightarrow \quad x_\mu^* \to x^*$$

Dual smoothing

Regularize the primal problem.

$$\underset{x}{\text{minimize}} \quad f(x) + \tfrac{\mu}{2}\|x\|^2 + g(Ax)$$

Dualize. Regularizing f smoothes the conjugate:

$$(f + \tfrac{\mu}{2}\|\cdot\|^2)^* = (f^* \,\square\, \tfrac{1}{2\mu}\|\cdot\|^2) = f^*_\mu$$

$$\underset{y}{\text{minimize}} \quad f^*_\mu(A^T y) + g^*(-y)$$

Example (Basis Pursuit).

$$f = \|\cdot\|_1, \qquad g = \delta_{\{b\}} \quad \Longrightarrow \quad \min_x \quad \|x\|_1 + \tfrac{\mu}{2}\|x\|^2 \quad \text{subj to} \quad Ax = b$$

$$f^*_\mu = \tfrac{1}{2\mu}\,\mathrm{dist}^2_{\mathbb{B}_\infty} \quad \Longrightarrow \quad \max_y \quad \langle b, y\rangle + \tfrac{1}{2\mu}\,\mathrm{dist}^2_{\mathbb{B}_\infty}(A^T y)$$

Exact regularization: $\quad x^*_\mu = x^*$ for all $\mu \in [0, \bar{\mu})$, $\quad \bar{\mu} > 0$

[Mangasarian and Meyer (1979); Friedlander and Tseng (2007)]

[Nesterov (2005); Beck and Teboulle (2012)]; TFOCS [Becker et al. (2011)]

CONCLUSION

much left unsaid, eg,

- optimal first-order methods [Nesterov (1988, 2005)]
- block coordinate descent [Sardy et al. (2004)]
- active-set methods (eg, LARS & Homotopy) [Osborne et al. (2000)]

my own work.

- avoiding proximal operators [Friedlander et al. (2014) (SIOPT)]
- greedy coordinate descent [Nutini et al. (2015) (ICML)]

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