# CONVEX SETS

FEB 18, 2018 (1st ver)

FEB 24, 2020 (revised)

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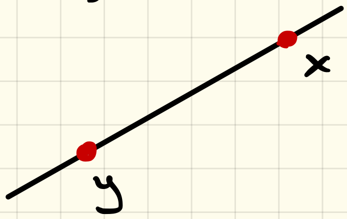
# AFFINE SETS

A Line through the distinct points $x, y \in \mathbb{R}^n$:

$$\{ z \mid \theta x + (1-\theta) y = z \ , \ \theta \in \mathbb{R} \}$$




An affine set contains all lines through any 2 distinct points in the set

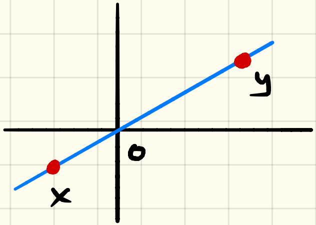
Example the solution set of linear equations

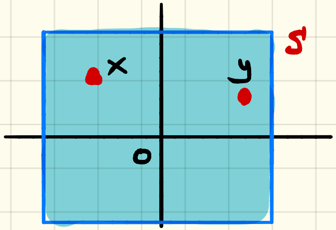
$$S = \{ x \mid Ax = b \}$$

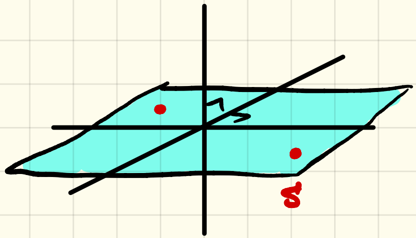
Pf: $x_1, x_2 \in S \Rightarrow Ax_1 = Ax_2 = b \Rightarrow A(\theta x_1 + (1-\theta) x_2) = \theta b + (1-\theta) b = b.$ //

# Linear Sets

$S$ is the subspace containing distinct points $x, y \in \mathbb{R}^n$ and the origin:



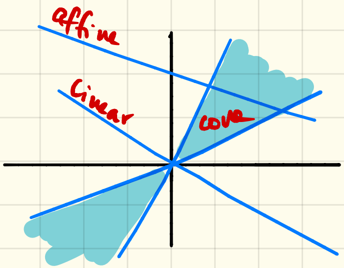

$$S = \{ z \mid \alpha x + \beta y = z \,,\ \alpha, \beta \in \mathbb{R} \} \qquad (\alpha = \beta = 0 \text{ included})$$

The set $S$ is linear if it contains all lines through any distinct points in the set

$$x \in S \,,\ y \in S \iff z : \alpha x + \beta y \in S \quad \forall\ \alpha, \beta \in \mathbb{R}$$

Examples: range and null of a matrix $A$

$$\text{range}(A) = \{ y \in \mathbb{R}^m \mid y = Ax,\ x \in \mathbb{R}^n \}$$
$$\text{null}(A) = \{ z \in \mathbb{R}^n \mid Az = 0 \}$$



# Convex Sets

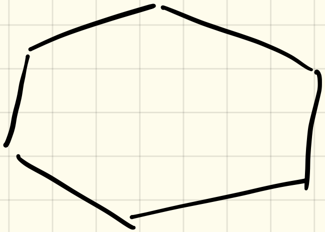
The <u>line segment</u> between any two points $x, y$ :

$$z = \theta x + (1-\theta) y \quad \forall \ \theta \in [0,1]$$

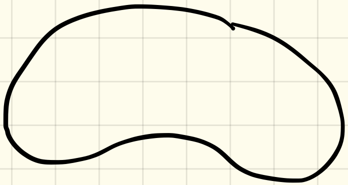
A set $C \subseteq \mathbb{R}^n$ is <u>convex</u> if it contains the line segment between any two distinct points in the set:

$$x, y \in C \ , \quad 0 \leq \theta \leq 1 \implies \theta x + (1-\theta) y \in C$$

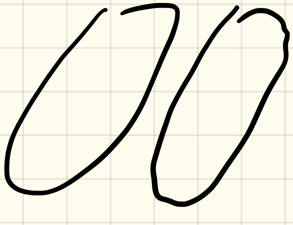
<u>Examples</u>



convex



non-convex



non-convex

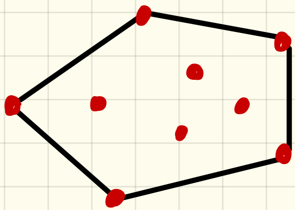
# CONVEX COMBINATIONS AND HULLS

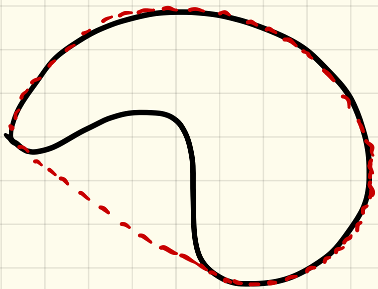
Convex combination of a set of points $x_1, \cdots, x_K$:

$$x = \theta_1 x_1 + \cdots + \theta_K x_K$$

with $\sum_{i=1}^{K} \theta_i = 1$, $\theta_i \geq 0$. (eg, $\theta \in \Delta_K$)

The convex hull of a set $D$ is the set that contains all convex combinations of points in $D$

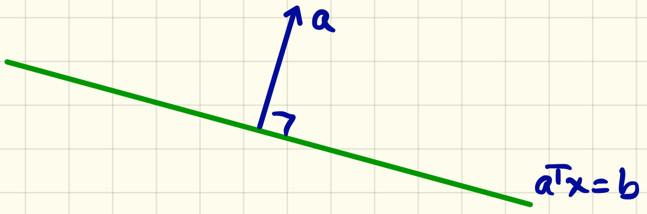




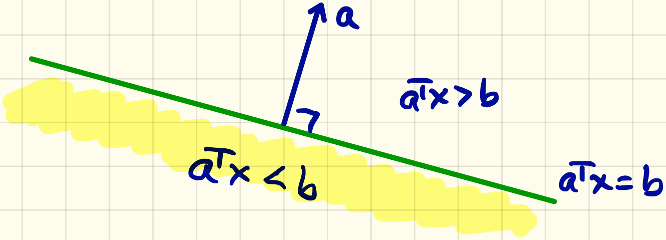
$$\mathrm{co}\, S = \{ z = \theta x + (1-\theta) y \mid x, y \in S, \ \theta \in [0,1] \}$$

# Half Spaces and Hyperplanes

<u>hyperplane</u>: sets of the form $\{x \mid a^T x = b\}$ with $a \neq 0$.




<u>halfspace</u>: sets of the form $\{x \mid a^T x \leq b\}$ with $a \neq 0$.




- $a$ is the "normal" vector
- hyperplanes are affine <u>and</u> convex; halfspaces are convex
- halfspaces are convex but <u>not</u> affine.

# Operations that Preserve Convexity I

Intersections : Let $C_i \subseteq \mathbb{R}^n$ be convex sets $(i \in I)$. Then

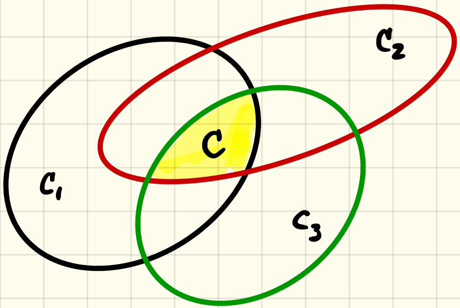
$$\bigcap_{i \in I} C_i \quad \text{is convex}$$

Proof : Suppose $x, y \in \bigcap_{i \in I} C_i$ and let $\lambda \in [0,1]$.

$\Rightarrow \quad x, y \in C_i \qquad \forall i \in I$

$\Rightarrow \quad \lambda x + (1-\lambda) y \in C_i \qquad \forall i \in I$, because $C_i$ cvx

$\Rightarrow \quad \lambda x + (1-\lambda) y \in \bigcap_{i \in I} C_i$



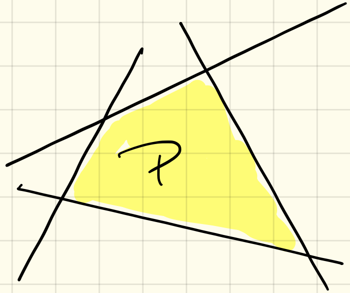
# Convex Polytopes

A set defined by a set of linear inequalities is convex:

$$P = \{x \in \mathbb{R}^n \mid Ax \leq b\} \qquad A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix} \ (m \times n)$$

$$= \bigcap_{i=1\ldots m} \{x \mid a_i^T x \leq b_i\}$$

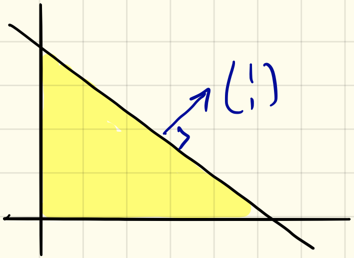
$P$ is a <u>convex polytope</u>.




<u>Example</u> n-dimensional simplex

$$\{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i \leq 1 \ , \ x_i \geq 0\}$$

$$= \{x \in \mathbb{R}^n \mid e^T x \leq 1\} \cap \{x \mid -e_1^T x \leq 0\}$$
$$\vdots$$
$$\cap \{x \mid -e_n^T x \leq 0\}$$



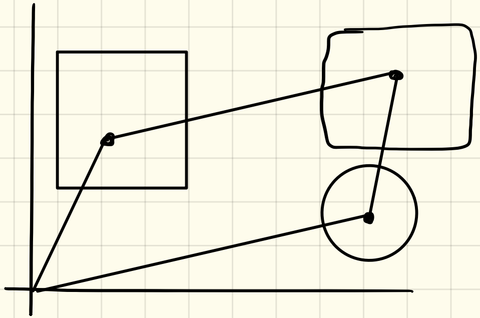
# Operations that Preserve Convexity II

<u>Linear Mapping</u> : Let $C \subseteq \mathbb{R}^n$ be cvx. For $A$ $m \times n$, the image of $C$ under $A$ is convex :

$$A(C) = \{ Ax \mid x \in C \}$$

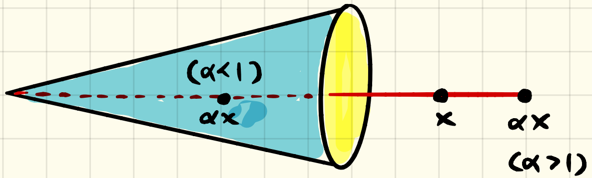
<u>Addition</u> : Let $C_1, \cdots, C_m \subseteq \mathbb{R}^n$ be cvx. Then the addition of sets is convex :

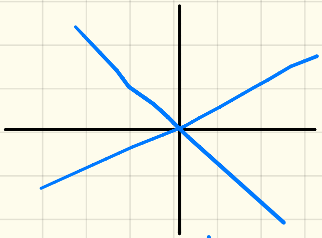
$$\lambda_1 C_1 + \lambda_2 C_2 + \cdots + \lambda_m C_m = \{ \sum \lambda_i x_i \mid x_i \in C_i \quad i = 1, \cdots, m \}$$



# Convex Cones

A set $S \subseteq \mathbb{R}^n$ is a <u>cone</u> if $x \in S \Leftrightarrow \alpha x \in S \quad \forall\, \alpha \geq 0$.




A convex cone is a cone that is convex

$$x, y \in S \Leftrightarrow \theta_1 x + \theta_2 y \in S \quad \forall\, \theta_1, \theta_2 \geq 0$$

Examples of convex cones:

$$\mathbb{R}^n_+ = \{ x \mid x_j \geq 0 \ \forall j = 1:n \}$$ non-negative orthant

$$L^n_+ = \left\{ \begin{bmatrix} x \\ t \end{bmatrix} \in \mathbb{R}^{n+1} \;\middle|\; \|x\|_2 \leq t \quad x \in \mathbb{R}^n, \ t \in \mathbb{R}_+ \right\}$$ second-order cone

$$S^n_+ = \{ X \in \mathbb{R}^{n \times n} \mid u^T X u \geq 0 \ \forall u \in \mathbb{R}^n, \ X = X^T \}$$ positive semi-def cone

EXAMPLE The polytope $P = \{x \mid Ax \leq 0\}$ is a cone

$$x \in P,\ \lambda \geq 0 \Rightarrow Ax \leq 0,\ \lambda \geq 0$$
$$\Rightarrow A(\lambda x) \leq 0 \Rightarrow \lambda x \in P \quad \forall \lambda \geq 0.$$

EXAMPLE The Lorentz cone

$$L^n = \left\{ \begin{pmatrix} x \\ \alpha \end{pmatrix} \in \mathbb{R}^{n+1} \;\middle|\; \|x\| \leq \alpha,\ x \in \mathbb{R}^n,\ \alpha \in \mathbb{R} \right\}$$