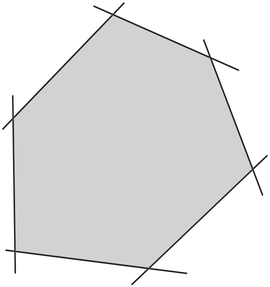
Geometry of Linear Programming

- extreme points
- vertices
- basic (feasible) solutions

Polyhedron (inequality form)

$A = \begin{bmatrix} a_1 & \cdots & a_m \end{bmatrix}^T$ is $m \times n$, $\quad b \in \mathbb{R}^m$

$$\mathcal{P} = \{ x \mid Ax \le b \} = \left\{ x \;\middle|\; a_i^T x \le b_i, \; i = 1, \ldots, m \right\}$$

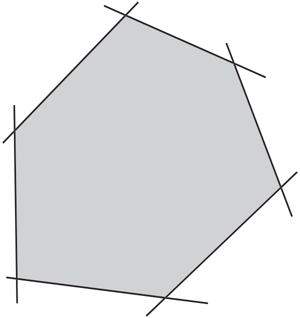


$\mathcal{P}$ is convex because it's the intersection of halfspaces (the intersection of convex sets is convex)

Extreme points

$x \in \mathcal{P}$ is an **extreme point** of $\mathcal{P}$ if there **does not exist** two vectors $y, z \in \mathcal{P}$ such that

$$x = \lambda y + (1 - \lambda) z \quad \text{for any} \quad \lambda \in (0, 1)$$



Vertices

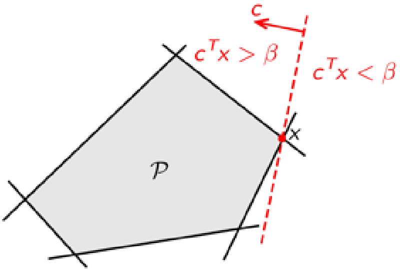
$x \in \mathcal{P}$ is a **vertex** of $\mathcal{P}$ if there exists a vector $c \neq 0$ such that

$$c^T x < c^T y \quad \text{for all} \quad y \in \mathcal{P},\ y \neq x$$

two equivalent points of view:

- given a **vertex** x, find c such that $c^T x < c^T y$ for all $y \in \mathcal{P}$, $y \neq x$
- given a **vector** c, find x such that $c^T x < c^T y$ for all $y \in \mathcal{P}$, $y \neq x$, ie,

$$\underset{x}{\text{minimize}} \quad c^T x \quad \text{subject to} \quad x \in \mathcal{P}$$



Active constraints

define $\mathcal{B}$ as the set of **active** or **binding** constraints (at x^*):

$$
\begin{aligned}
a_i^T x^* &= b_i, \quad i \in \mathcal{B} && \text{(active constraints)} \\
a_i^T x^* &< b_i, \quad i \in \mathcal{N} && \text{(inactive feasible constraints)} \\
a_i^T x^* &> b_i, \quad i \notin \mathcal{B} \cup \mathcal{N} && \text{(inactive infeasible constraints)}
\end{aligned}
$$

define the subset of active constraints

$$
A_{\scriptscriptstyle B} = \bar{A} = \begin{bmatrix} a_{i_1}^T \\ a_{i_2}^T \\ \vdots \\ a_{i_k}^T \end{bmatrix}, \qquad b_{\scriptscriptstyle B} = \bar{b} = \begin{bmatrix} b_{i_1} \\ b_{i_2} \\ \vdots \\ b_{i_k} \end{bmatrix}, \qquad \mathcal{B} = \{i_1, i_2, \ldots, i_k\}
$$

Basic solutions

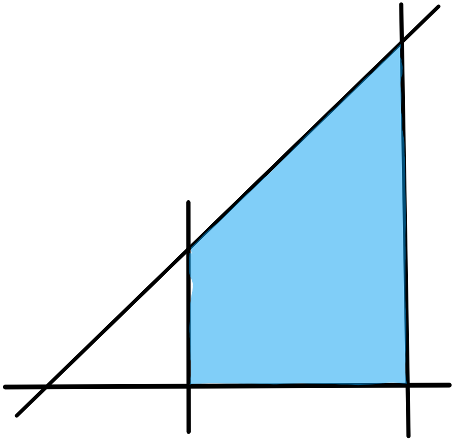
x^* is a **basic solution** if one of the following equivalent conditions hold:

- $a_{i_1}, a_{i_2}, \ldots, a_{i_n}$ are linearly independent
- $\bar{A}x^* = \bar{b}$ has a unique solution
- $\text{rank}(\bar{A}) = n$

basic feasible solution: x^* is a basic solution and $x^* \in \mathcal{P}$

Theorem: the following are equivalent

- x^* is a vertex
- x^* is an extreme point
- x^* is a basic feasible solution



Example

$$\begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} x \leq \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \end{bmatrix}$$

- $(1,1)$ is an extreme point
- $(1,1)$ is a vertex: unique minimum of $c^T x$ with $c = (-1,-1)$
- $(1,1)$ is a basic feasible solution: $\mathcal{B} = \{2,4\}$ and rank $\bar{A} = 2$, where

$$\bar{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Equivalence of definitions

Vertex $\Longrightarrow$ extreme point

Let x^* be a vertex of $\mathcal{P}$. Then there exists a $c \neq 0$ such that

$$c^T x^* < c^T x \quad \text{for all } x \in \mathcal{P} \text{ and } x \neq x^*.$$

Then for all $y, z \in \mathcal{P}$ with $y \neq x^*$ and $z \neq x^*$,

$$c^T x^* < c^T y \quad \text{and} \quad c^T x^* < c^T z.$$

If $\lambda \in [0, 1]$, then

$$c^T x^* < c^T (\lambda y + (1 - \lambda) z),$$

and $x^* \neq \lambda y + (1 - \lambda) z$. Therefore x^* is an extreme point.

Extreme point $\Longrightarrow$ basic feasible solution

Suppose $x^* \in \mathcal{P}$ is an extreme point with

$$a_i^T x^* = b_i \quad i \in \mathcal{B}, \qquad \text{and} \qquad a_i^T x^* < b_i \quad i \notin \mathcal{B}.$$

Proceed by contradiction. Suppose x^* is not a basic feasible solution. Thus, a_i for $i \in \mathcal{B}$ are not linearly independent. Then there exists a $d \neq 0$ with

$$a_i^T d = 0 \qquad \text{for every} \quad i \in \mathcal{B} \qquad (\text{ie, } \bar{A}d = 0)$$

and for $\epsilon > 0$ small enough,

$$y = x^* + \epsilon d \in \mathcal{P}, \quad z = x^* - \epsilon d \in \mathcal{P}.$$

Summing, we have

$$x^* = \tfrac{1}{2} y + \tfrac{1}{2} z,$$

which contradicts the assumption that x^* is an extreme point.

Basic feasible solution $\Longrightarrow$ vertex

Suppose $x^* \in \mathcal{P}$ is a basic feasible solution and

$$a_i^T x^* = b_i \quad i \in \mathcal{B}, \qquad \text{and} \qquad a_i^T x^* < b_i \quad i \notin \mathcal{B}.$$

Take any $x \in \mathcal{P}$. For each $i \in \mathcal{B}$,

$$-a_i^T x \geq -b_i = -a_i^T x^*$$

Summing these all together:

$$c^T x = -\sum_{i \in \mathcal{B}} a_i^T x \geq -\sum_{i \in \mathcal{B}} b_i = c^T x^*, \qquad c := -\sum_{i \in \mathcal{B}} a_i$$

with equality only if $a_i^T x = b_i, i \in \mathcal{B}$. Since $\{a_i \mid i \in \mathcal{B}\}$ are linearly independent, that holds only when $x = x^*$. Thus, $c^T x^* < c^T x$ for all $x \in \mathcal{P}$, $x \neq x^*$, so x^* is a vertex.

Unbounded directions

$\mathcal{P}$ contains a **half-line** if there exists $d \neq 0$, x_0 such that

$$x_0 + \alpha d \in \mathcal{P} \quad \text{for all} \quad \alpha \geq 0$$

equivalent conditions for $\mathcal{P} = \{ x \mid Ax \leq b \}$:

$$Ax_0 \leq b, \quad Ad \leq 0$$

fact: $\mathcal{P}$ unbounded $\Longleftrightarrow$ $\mathcal{P}$ contains a half line

$\mathcal{P}$ contains a **line** if there exists $d \neq 0$, x_0 such that

$$x_0 + \alpha d \in \mathcal{P} \quad \text{for all} \quad \alpha$$

equivalent conditions for $\mathcal{P} = \{ x \mid Ax \leq b \}$:

$$Ax_0 \leq b, \quad Ad = 0$$

fact: $\mathcal{P}$ has no extreme points $\Longleftrightarrow$ $\mathcal{P}$ contains a line

Optimal set of an LP

$$\text{minimize} \quad c^T x \quad \text{subject to} \quad Ax \le b$$

- optimal value $p^* = \min \left\{ c^T x \;\middle|\; Ax \le b \right\}$ $\quad (p^* = \pm\infty$ is possible)
- optimal point: x^* with $Ax^* \le b$ and $c^T x^* = p^*$
- optimal set: $X^* = \left\{ x \;\middle|\; Ax \le b,\ c^T x = p^* \right\}$

example

$$\begin{array}{ll} \text{minimize} & c_1 x_1 + c_2 x_2 \\ \text{subject to} & -2x_1 + x_2 \le 1, \quad x_1 \ge 0,\ x_2 \ge 0 \end{array}$$

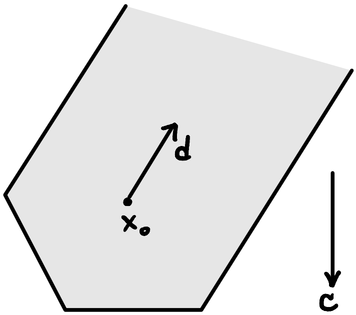
- $c = (1,1)$: $X^* = \{ (0,0) \}$, $p^* = 0$
- $c = (1,0)$: $X^* = \{ (0, x_2) \mid 0 \le x_2 \le 1 \}$, $p^* = 0$
- $c = (-1,-1)$: $X^* = \emptyset$, $p^* = -\infty$

Optimal values

- $p^* = -\infty$ if and only if there exists a feasible half line

$$\{\, x_0 + \alpha p \mid \alpha \geq 0 \,\}$$

with $c^T p < 0$



- $p^* = +\infty$ if and only if $\mathcal{P} = \emptyset$
- p^* if finite if and only if $X^* \neq \emptyset$

LP solutions on extreme points

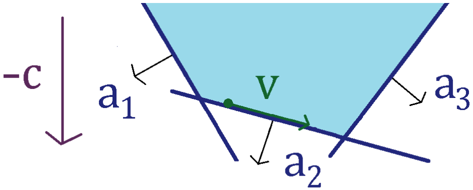
$$p^* = \min_{x \in \mathbb{R}^n} \{ c^T x \mid Ax \le b \}$$

if p^* finite, there exists a feasible extreme point x^* with $c^T x^* = p^*$

suppose $\hat{x}$ is optimal but not extreme. Then corresponding A_B (active rows of A) has a nontrivial nullspace, and there exists $d \ne 0$ such that $A_B d = 0$ and either

$$c^T d = 0 \quad \text{or} \quad c^T d < 0 \quad \text{or} \quad c^T d > 0$$

- suppose $c^T d < 0$
- pick $\tilde{x} = \hat{x} + \alpha d$
- then $c^T \hat{x} > c^T \tilde{x}$ and $\bar{A}_B \tilde{x} = A_B(\hat{x} + \alpha d) = A_B \hat{x}$
- for α small enough, $A_N \hat{x} < b_N \Rightarrow A_N \tilde{x} \leq b_N$
- then $\tilde{x}$ is feasible with lower objective value, contradicting optimality of $\hat{x}$



- the case with $c^T d > 0$ is similar, except we pick $\tilde{x} = \hat{x} - \alpha d$

- Suppose $c^T d = 0$. Then any adjacent extreme point is equally optimal
- pick $\tilde{x} = \hat{x} + \alpha d$
- then $c^T \hat{x} = c^T \tilde{x}$ and $A_B \hat{x} = A_B \tilde{x}$
- pick α small enough, $A_N \hat{x} < b_N \Rightarrow A_N \tilde{x} \leq b_N$
- then $\tilde{x}$ is feasible with same objective value, and there are infinitely many solutions on that edge