LP/QP Interior Algorithms

- barrier function
- primal barrier method
- perturbed optimality conditions
- Newton's method
- primal-dual method for LPs and QPs

Interior-point algorithms

The Simplex algorithm:

- "walks" the edges of the polyhedral feasible set
- worst-case complexity is exponential (may need to visit **every** vertex)
- experience (and some analysis) suggests average polynomial complexity

Interior-point (IP) are a radical departure from the simplex method:

- IP algorithms traverse the interior of the polyhedral set
- (impractical) polynomial algorithm for LP first proposed by Kachian (1979)
- Karmarkar (1984) offered first "practical" polynomial LP algorithm
    - AT&T wouldn't release details
    - patented the KORBYX, a computer that implemented the method
    - appeared in front-page of New York Times

Eliminate nonnegativity constraints

Apply to the primal LP problem in standard form:

$$\underset{x}{\text{minimize}} \quad c^T x \quad \text{subject to} \quad Ax = b,\ x \geq 0$$

The core difficulty in LP is the prescence of the constraint $x \geq 0$

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Eliminate nonnegativity constraint via **barrier function**:

$$B_\mu(x) = c^T x - \mu \sum_j \log x_j$$

- $-\log x_j \to \infty$ as $x_j \to 0^+$ (def'd as $+\infty$ for $x_j \leq 0$)
- $-\mu \sum_j \log x_j \to \infty$ as any $x_j \to 0^+$

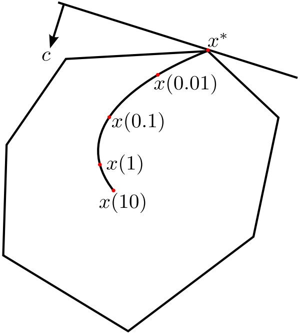
Barrier function

$$(P_\mu) \qquad \underset{x}{\text{minimize}} \quad B_\mu(x) \quad \text{subject to} \quad Ax = b$$

- minimizer of the barrier problem depends on μ:

$$x_\mu \quad \text{solves} \quad P_\mu$$

- minimizer of P_μ is unique for each μ because of convexity of B_μ



Example 1: $\underset{x}{\text{minimize}} \quad x \quad \text{subj to} \quad x \geq 0$

$$B_\mu(x) = x - \mu \log x \quad \Longrightarrow \quad x(\mu) = \mu$$

Example 2: $\underset{x_1,x_2,x_3}{\text{minimize}} \quad x_2 \quad \text{subj to} \quad x_1 + x_2 + x_3 = 1,\ x \geq 0$

$$B_\mu(x) = x_2 - \mu \log(x_1) - \mu \log(x_2) - \mu \log(x_3)$$

Eliminate $x_3 = 1 - x_1 - x_2$:

$$\underset{x_1,x_2}{\text{minimize}} \quad x_2 - \mu \log(x_1) - \mu \log(x_2) - \mu \log(1 - x_1 - x_2)$$

- $x_1(\mu) = \frac{1 - x_2(\mu)}{2}$
- $x_2(\mu) = \frac{1 + 2\mu - \sqrt{1 + 9\mu^2 + 2\mu}}{2}$
- $x_3(\mu) = \frac{1 - x_2(\mu)}{2}$

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This problem has infinitely many solutions:

$$X^* = \{\, x \mid x = (x_1, 0, x_3),\ x_1 + x_3 = 1, x \geq 0 \,\}$$

Note that the solution that we converge to isn't basic.

Primal barrier method

solve a sequence of linearly constrained **nonlinear** functions:

choose $x_0 > 0$, $\mu_0 > 0 (\approx 1)$, $\tau < 1$
repeat

x_{k+1} minimizes $B_{\mu_k}(x)$ subj to $Ax = b$
$\mu_{k+1} \leftarrow \tau \mu_k$

until μ_k is "small"

under mild conditions, $x_k \to x^*$

Perturbed optimality conditions

primal LP:

minimize $c^T x - \mu \sum_j \log x_j$
subject to $Ax = b$

Optim cond's:

$$c + \mu X^{-1} e = A^T y$$

$$Ax = b \ (x > 0)$$

dual LP:

maximize $b^T y + \mu \sum_j \log z_j$
subject to $A^T y + z = c$

Optim cond's:

$$Aw = -b$$

$$-\mu Z^{-1} e = w$$

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Tie these optimality cond's together by identifying $x \equiv -w$ and noting

$$\mu Z^{-1} e = x \quad \Longleftrightarrow \quad \mu \frac{1}{z_j} = x_j \quad \Longleftrightarrow \quad x_j z_j = \mu$$

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write both optimality conditions simultaneously as

$$\begin{aligned} A^T y + z &= c, \quad z > 0 \\ Ax &= b, \quad x > 0 \\ x_j z_j &= \mu, \quad j = 1, \ldots, n \end{aligned}$$

Newton's method

$$F_\mu(x,y,z) = \begin{bmatrix} Ax - b \\ A^T y + z - c \\ Xz - \mu e \end{bmatrix}$$

An approximate LP solution (x, y, z), with $(x, z) > 0$ satisfies

$$F_\mu(x,y,z) = 0$$

Apply Newton's method for root finding to these equations, eg,

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} + \alpha \begin{bmatrix} p_x \\ p_y \\ p_y \end{bmatrix}$$

where p is a Newton step:

$$J_k p = -F_k \quad \iff \quad \begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ Z_k & 0 & X_k \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} b - Ax_k \\ c - A^T y_k - z_k \\ \mu e - X_k z_k \end{bmatrix}$$

Primal-dual method for LPs

choose $x_0 > 0$, y_0, $z_0 > 0$, $\tau < 1$
$\gamma_0 \leftarrow x_0^T z_0$, $k \leftarrow 0$
while $\gamma_k > \epsilon$ **do**

$\mu_k = \tau(x_k^T z_k)/n$

Solve $J_k p = -F_k$ for $p = (p^x, p^y, p^z)$ [Newton step]

$$\beta_k^x = \min\left\{1,\ .995 \min_{\{j|p_j^x<0\}} -\frac{x_j^k}{p_j^x}\right\}$$

$$\beta_k^z = \min\left\{1,\ .995 \min_{\{j|p_j^z<0\}} -\frac{z_j^k}{p_j^z}\right\}$$

$x_{k+1} \leftarrow x_k + \beta_k^x p^x$
$y_{k+1} \leftarrow y_k + \beta_k^z p^y$
$z_{k+1} \leftarrow z_k + \beta_k^z p^z$

$k \leftarrow k + 1$

end

Linear Algebra

The main work is in computing the step directions (p_x, p_y, p_z) via

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It's common to eliminate p_z and solve the block 2-by-2 system

$$\begin{bmatrix} -X_k^{-1} Z_k & A^T \\ A & \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} = \begin{bmatrix} r_p \\ r_d - X_k^{-1} r_\mu \end{bmatrix}$$

Quadratic Programming (QP)

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^T Q x + c^T x \\ \text{subject to} & Ax = b,\ x \geq 0 \end{array}$$

This is a much more general problem than LP. (Setting $Q = 0$ gives an LP.)

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Example: The "distance" between two polyhedra

$$\mathcal{P}_1 = \{\, x \mid A_1x \leq b_1 \,\} \qquad \text{and} \qquad \mathcal{P}_2 = \{\, x \mid A_2x \leq b_2 \,\}$$

is defined by the solution of the quadratic program

$$\begin{array}{ll} \underset{x_1, x_2}{\text{minimize}} & \frac{1}{2}\|x_1 - x_2\|_2^2 \\ \text{subject to} & A_1x_1 \leq b_1 \quad \text{and} \quad A_2x_2 \leq b_2 \end{array}$$

Primal-dual approach for QPs

QP:

minimize $\frac{1}{2}x^TQx + c^Tx - \mu \sum_j \log x_j$
subject to $Ax = b$

Optim cond's:

$$Qx + c + \mu X^{-1}e = A^Ty$$
$$Ax = b \; (x > 0)$$

LP:

minimize $c^Tx - \mu \sum_j \log x_j$
subject to $A^Ty + z = c$

Optim cond's:

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Optim cond's:

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$$Ax = b \; (x > 0)$$

Define z such that $x_jz_j = \mu$. The optimality conditions become

$$\begin{aligned} -Qx + A^Ty + z &= c, \quad z > 0 \\ Ax &= b, \quad x > 0 \\ x_jz_j &= \mu, \quad j = 1, \ldots, n \end{aligned}$$

$$F_\mu(x, y, z) = \begin{bmatrix} Ax - b \\ -Qx + A^Ty + z - c \\ Xz - \mu e \end{bmatrix}$$

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