Simplex: Details

- General bounds: $\ell \leq x \leq u$
- Two-phase simplex

General upper and lower bounds

standard form

$$\begin{array}{ll} \underset{x}{\text{minimize}} & c^T x \\ \text{subject to} & Ax = b,\ x \geq 0 \end{array}$$

general bounds

$$\ell \leq x \leq u$$

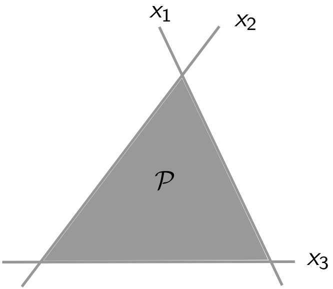
reduction to standard form

$$\begin{array}{rlcl} x - s_1 = \ell, & s_1 \geq 0 & \Longrightarrow & \ell \leq x \\ x + s_2 = u, & s_2 \geq 0 & \Longrightarrow & x \geq u \end{array}$$

standard form problem

$$\begin{array}{ll} \underset{x, s_1, s_2}{\text{minimize}} & c^T x \\ \text{subject to} & \begin{bmatrix} A & & \\ I & -I & \\ I & & I \end{bmatrix} \begin{bmatrix} x \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} b \\ \ell \\ u \end{bmatrix}, \quad \begin{bmatrix} x \\ s_1 \\ s_2 \end{bmatrix} \geq 0 \end{array}$$

General bounds and nonbasic variables



- nonbasic variables are always at their bounds
- basic variables uniquely determined by nonbasic variables

$$b = Ax = \begin{bmatrix} B & N \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} = Bx_B + Nx_N$$

thus,

$$Bx_B = b - Nx_N, \qquad (x_N)_j = \ell_j \quad \text{or} \quad (x_N)_j = u_j, \qquad j = \eta_1, \ldots, \eta_{n-m}$$

Simplex with General Bounds

effect on objective: need to choose a "good" d_N. Solve

$$B^T y = c_B \qquad \text{and} \qquad z := c - A^T y$$

$$\bar{\phi} = \phi + \alpha z_N^T d_N$$

pricing: only one nonbasic η_p moves, implying

$$d_N = \pm e_p, \quad B d_B = -a_{\eta_p}$$

choose p so that

$$\begin{array}{llll} \ell_{\eta_p} = x_{\eta_p} & \text{and} & z_{\eta_p} < 0 & \Longrightarrow \text{ set } d_{\eta_p} = \;\;\, 1 \\ u_{\eta_p} = x_{\eta_p} & \text{and} & z_{\eta_p} > 0 & \Longrightarrow \text{ set } d_{\eta_p} = -1 \end{array}$$

optimality: no improving direction exists if for each $j = 1, \ldots, n$

$$\begin{array}{ccccccc} \ell_j & = & x_j & & & \text{and} & z_j \geq 0 \\ & & x_j & = & u_j & \text{and} & z_j \leq 0 \\ \ell_j & \leq & x_j & \leq & u_j & \text{and} & z_j = 0 \end{array}$$

Ratio Test for General Bounds

with simple bounds,

- basic variable leaves basis
- nonbasic variable enters basis (use ratio test to determine)

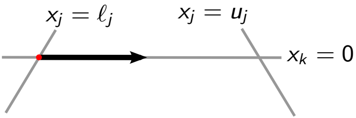
with general bounds, a nonbasic variable may both "enter and leave", eg,

- $\ell_j = x_j$ increases and first variable to hit bound (u_j)
- $u_j = x_j$ decreases and first variable to hit bound (ℓ_j)

in this case, basis does not change, but x_B does:

$$Bx_B = b - Nx_N$$

where some component of x_N changes



Finding an Initial Basic Feasible Solution

not always obvious how to choose a linearly independent set of columns B so that $x \geq 0$, where

$$\begin{bmatrix} B & N \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} = b, \quad Bx_B = b - Nx_N = b$$

auxiliary linear program

$$\begin{array}{ll} \underset{x,s}{\text{minimize}} & e^T s \\ \text{subject to} & Ax + s = b, \quad x, s \geq 0 \end{array}$$

the variables s are "artificial"

Two-phase Simplex

Phase 1:

1. ensure that $b \geq 0$
2. apply Simplex to auxiliary problem with

$$\bar{A} = \begin{bmatrix} A & I \end{bmatrix}, \quad \bar{x} = (x, s), \quad \bar{x} \geq 0, \quad \mathcal{B} = \{\, n+1, n+2, \ldots, n+m \,\}$$

3. if solution $\bar{x}^* = (x^*, s^*)$ has $s^* \neq 0$, original LP is infeasible. **Stop**.
4. if $s^* = 0$, then original LP is feasible.
5. optimal basis will not have any auxiliary variables (assuming no degeneracy)

Phase 2:

1. use optimal basis from Phase 1 as initial BFS
2. apply Simplex to original LP