Standard Form Polyhedra

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Polyhedra in standard form

generic polyhedron

$$\mathcal{P} = \left\{ x \;\middle|\; \begin{array}{r} Ax = b \\ Cx \leq d \end{array} \right\}$$

standard-form polyhedron

$$\mathcal{P} = \left\{ x \;\middle|\; \begin{array}{r} Ax = b \\ x \geq 0 \end{array} \right\} \qquad \text{with} \qquad b \geq 0$$

Converting to standard form: positive b

For $b_i < 0$, replace

$$a_i x = b_i \quad \longrightarrow \quad (-a_i)x = (-b_i)$$

For $d_i < 0$, replace

$$\begin{aligned} c_i^T x \leq d_i \quad &\longrightarrow \quad (-c_i)^T x \geq (-d_i) \\ c_i^T x \geq d_i \quad &\longrightarrow \quad (-c_i)^T x \leq (-d_i). \end{aligned}$$

Converting to standard form: free variables

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Converting free variables

- every free variable x_i is replaced with two new variables x_i' and x_i'', ie,

$$x_i := x_i' - x_i'', \quad x_i' \geq 0 \quad \text{and} \quad x_i'' \geq 0$$

- x_i' encodes the positive part of x_i
- x_i'' encodes the negative part of x_i
- optimal solution must have $x_i' \cdot x_i'' = 0$

Converting to standard form: slack and surplus

For every inequality constraint of the form

$$c_i^T x \leq d_i \qquad \left(c_i^T x \geq d_i\right)$$

introduce a new **slack** (or **surplus**) variable s_i, replacing the inequality with two constraints

$$\begin{aligned} c_i^T x + s_i &= d_i \\ s_i &\geq 0 \end{aligned} \qquad \left(\begin{aligned} c_i^T x - s_i &= d_i \\ s_i &\geq 0 \end{aligned}\right)$$

Basic solutions in standard form

x^* is a **basic solution** if the vectors

$$a_{i_1}, a_{i_2}, \ldots, a_{i_n}, \quad i_j \in \mathcal{B}$$

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- n variables $(x_1, \ldots, x_n)$
- $m + n$ total constraints
    - m equality constraints $(Ax = b)$
    - n inequality constraints $(x \geq 0)$

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for any basic solution x,

- the basic set $\mathcal{B}$ must have n elements
- thus, exactly n of the constraints need to be active at x
- m equality constraints are always satisfied
- thus $n - m$ of the inequality constraints $x \geq 0$ should be "active"

Basic solutions in standard form

Choosing $n - m$ of the inequality constraints to be active is the same as choosing $n - m$ variables x_i to be zero. Making x_i zero effectively eliminates column i from the matrix A.

This is equivalent to choosing m columns of A! To be a basic solution, we also need these m columns to be linearly independent. So, permute the variables and partition

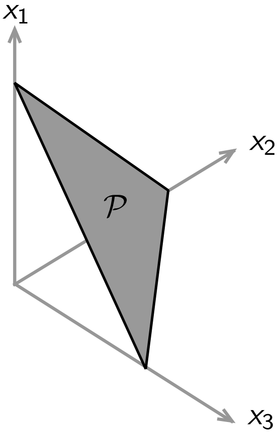
$$AP = [B \quad N] \qquad \text{where } B \text{ is nonsingular}$$

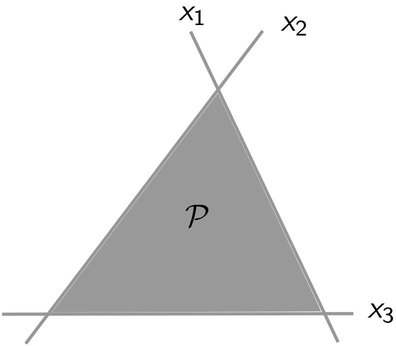
Now we have

$$\bar{A}x = \begin{bmatrix} B & N \\ & I \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

$$x_N = 0$$
$$Bx_B = b$$

Two-dimensional representation





Degeneracy: inequality form

polyhedron in inequality form:

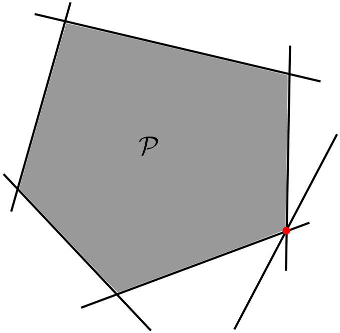
$$Ax \le b$$

a basic feasible solution x^* with

$$a_i^T x^* = b_i, \quad i \in \mathcal{B} \qquad \text{and} \qquad a_i^T x^* < b_i, \quad i \notin \mathcal{B}$$

is **degenerate** if # of indices in $\mathcal{B}$ is greater than n

- property of the **description** of the polyhedron
- affects the performance of some algorithms
- disappears for small perturbations of b



Degeneracy: standard form

polyhedron in standard form:

$$Ax = b, \quad x \geq 0$$

a basic solution partitions the variables into two sets:

$$\begin{bmatrix} B & N \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} = b \qquad \text{with} \qquad x_N = 0$$

ie,

$$Bx_B = b$$

a basic feasible solution in standard form is **degenerate** if more than $n - m$ components in x are zero, ie,

$$x = \begin{bmatrix} x_B \\ x_N \end{bmatrix} \begin{array}{l} m \\ n-m \end{array} = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix} \longleftarrow \text{has some zero components}$$