# Duality

- dual LP
- weak duality
- strong duality
- complementarity

## Duality

Consider the constrained problem

$$\begin{array}{ll} \underset{x_1,x_2}{\text{minimize}} & x_1^2 + x_2^2 \\ \text{subject to} & x_1 + x_2 = 1 \end{array}$$

and the unconstrained problem

minimize 
$$\phi(x_1, x_2, y) \equiv x_1^2 + x_2^2 + y(1 - x_1 - x_2)$$

The scalar y is the "price" for violating the constraint  $x_1 + x_2 = 1$ . What price y is enough to induce the optimal solution  $x^* = (1/2, 1/2)$ ?

$$\left. \begin{array}{l} \frac{\partial \phi}{\partial x_1} = 2x_1 - y = 0\\ \frac{\partial \phi}{\partial x_1} = 2x_2 - y = 0 \end{array} \right\} \Longrightarrow \quad x_1 = x_2 = \frac{y}{2} \quad \Longrightarrow \quad y^* = 1$$

## **Dual function**

**primal problem:** minimize  $c^T x$  subj to  $Ax = b, x \ge 0$ 

- *n* variables, *m* constraints
- optimal solution  $x^*$
- optimal value  $p^* \equiv c^T x^*$

**relaxed problem:** minimize  $c^T x + y^T (b - Ax)$  subj to  $x \ge 0$ 

- relaxed problem is a lower bound for *p*\*:
- $g(y) := \min_{x \ge 0} \{ c^T x + y^T (b Ax) \} \le c^T x^* + y^T (b Ax^*) = c^T x^* = p^*$

tightest lower bound: find y that solves

$$\max_{y} \max g(y)$$

### Dual of an LP

$$g(y) = \min_{x \ge 0} \{ c^T x + y^T (b - Ax) \}$$
$$= b^T y + \min_{x \ge 0} \{ x^T (c - A^T y) \}$$
$$= \begin{cases} b^T y & \text{if } c - A^T y \ge 0 \\ -\infty & \text{otherwise} \end{cases}$$

Because we want to **maximize** g(y), we must have

$$\begin{array}{lll} \underset{y}{\text{maximize}} & b^T y & \Longleftrightarrow & \underset{y,z}{\text{maximize}} & b^T y \\ \text{subject to} & c - A^T y \geq 0 & & \text{subject to} & A^T y + z = c \\ & & z \geq 0 \end{array}$$

this is the  $\ensuremath{\text{dual}}\xspace$  LP

### Weak duality

Suppose that x is primal feasible:

$$Ax = b, \quad x \ge 0$$

Suppose that (y, z) is dual feasible:

$$A^T y + z = c, \quad z \ge 0$$

Then the primal objective is bounded below by the dual objective:

$$c^{T}x = (A^{T}y + z)^{T}x = y^{T}Ax + z^{T}x = y^{T}b + \underbrace{z^{T}x}_{(\geq 0)} \ge y^{T}b$$

Weak-duality theorem: if (x, y, z) is primal/dual feasible, then

- the primal value is an upper bound for the dual value
- the dual value is a lower bound for the primal value

## Complementarity

primaldualminimize
$$c^T x$$
maximize  
 $y,z$  $b^T y$ subject to $Ax = b$ subject to  
 $x \ge 0$  $A^T y + z = c$ 

By weak duality:

(primal value) 
$$\equiv c^T x = b^T y + z^T x \ge b^T y \equiv (dual value)$$

This bound is "tight" when x and z are **complementary**, ie,  $x^T z = 0$ :

$$x_j = 0$$
 and  $z_j \ge 0$   
 $x_j \ge 0$  and  $z_j = 0$ 

## **Optimality conditions**

Simplex maintains primal feasibility at every iteration:

$$Ax = b, \quad x \ge 0$$

It defines y via  $B^T y = c_B$  and  $z = c - A^T y$ , and maintains complementarity:

$$egin{array}{lll} x_{\scriptscriptstyle B} \geq 0 & {
m and} & z_{\scriptscriptstyle B} = 0 & {
m (by \ construction)} \ x_{\scriptscriptstyle N} = 0 & {
m and} & z_{\scriptscriptstyle N} \lessapprox 0 \end{array}$$

Simplex exits when  $z \ge 0$ , ie, (y, z) is **dual feasible**, ie,

$$A^T y + z = c, \quad z \ge 0$$

**Strong duality theorem:** If an LP has an optimal solution, so does its dual, and the optimal values are equal, ie,  $p^* = d^*$ 

### Sufficient conditions

Suppose that (x, y, z) is primal/dual feasible.

By weak duality,

$$c^{T}x - b^{T}y = z^{T}x$$

By strong duality, if (x, y, z) is primal-dual optimal,

$$z^T x = 0$$

Conversely, if  $z^T x = 0$ , then

- $c^{T_X}$  achieves its lower bound
- $b^T y$  achieves its upper bound

therefore, (x, y, z) is primal-dual optimal

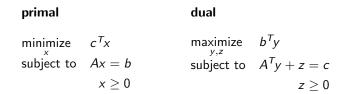
**Theorem:** The primal-dual triple (x, y, z) is optimal iff

$$Ax = b, x \ge 0, A^{T}y + z = c, z \ge 0, x^{T}z = 0$$

### Relationship between primal and dual LPs



#### Interpretation of dual variables



Suppose that  $x^*$  is optimal and nondegenerate, then

$$x^* = egin{bmatrix} B^{-1}b \ 0 \end{bmatrix} > 0 \qquad ext{and} \qquad x^*_{\scriptscriptstyle B}(\Delta b) = B^{-1}(b+\Delta b) > 0 \quad ext{for small } \|\Delta b\|$$

Reduced costs  $z^* = c - A^T y^*$  doesn't change. Thus  $(x^*(||\Delta b||), y^*, z^*)$  is primal/dual feasible