Convex Sets

CPSC 406 – Computational Optimization

Convex sets

- definition
- subspaces, affine sets, and spans
- halfspaces and hyperplanes
- cones and hulls
- operations that preserve convexity

Lines and line segments

- **line** through two points $x_1, x_2 \in \mathbb{R}^n$ is the set

$$\{z \mid z = \theta x_1 + (1-\theta) x_2,\ \theta \in \mathbb{R}\}$$

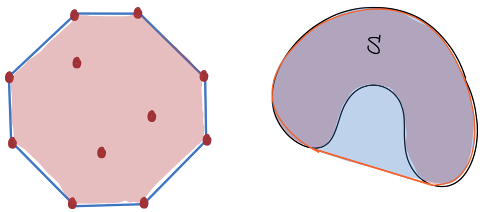
- $x \in \mathbb{R}^n$ is a **convex combination** of vectors $x_1, \ldots, x_k$ if

$$x = \sum_{i=1}^{k} \theta_i x_i, \quad \sum_{i=1}^{k} \theta_i = 1, \quad \theta_i \geq 0$$

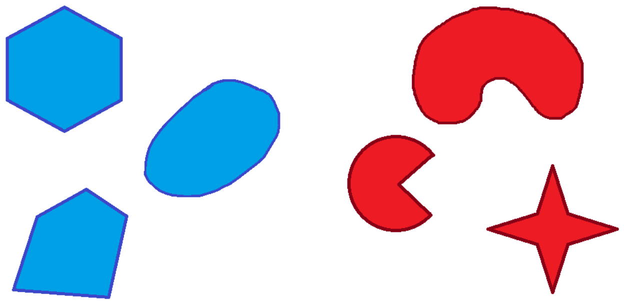
Convex sets and hulls

- the **convex hull** of a set of points $\mathcal{S}$ contains all convex combinations of points in $\mathcal{S}$:

$$\mathrm{conv}(\mathcal{S}) = \left\{ \sum_{i=1}^{k} \theta_i x_i \mid x_i \in \mathcal{S},\ \sum_{i=1}^{k} \theta_i = 1,\ \theta_i \geq 0 \right\}$$



- $\mathcal{C} \subset \mathbb{R}^n$ is **convex** if it contains all convex combinations of its elements, ie, $\mathcal{C} = \mathrm{conv}(\mathcal{C})$



Question

Show that the norm ball

$$\mathbb{B} = \{x \in \mathbb{R}^n \mid \|x - x_c\| \leq r\}$$

is a convex set.

Hint: use the triangle inequality

Subspaces, spans, and affine sets

- $\mathcal{S} \subset \mathbb{R}^n$ is a **subspace** if it contains all linear combinations of points in the set, ie,

$$\alpha x + \beta y \in \mathcal{S},\ \forall x, y \in \mathcal{S},\ \forall \alpha, \beta \in \mathbb{R}$$

- for any $m \times n$ matrix A, its **range** and **nullspace** are subspaces of $\mathbb{R}^n$:

$$\mathbf{range}(A) = \{Ax \mid x \in \mathbb{R}^n\} \quad \text{and} \quad \mathbf{null}(A^T) = \{z \mid A^T z = 0\}$$

- the **span** of a collection of vectors $x_1, \ldots, x_k$ is the subspace of all vectors

$$\mathbf{span}(x_1, \ldots, x_n) = \left\{ \sum_{i=1}^{k} \theta_i x_i, \quad \forall \theta_i \in \mathbb{R} \right\}$$

- an **affine set** is a translated a subspace, ie, for fixed $x_0 \in \mathbb{R}^n$ and subspace $\mathcal{S}$,

$$\mathcal{L} = \{x_0 + v \mid v \in \mathcal{S}\} \equiv x_0 + \mathcal{S}$$

- $\mathcal{S}$ is the **subspace parallel to** $\mathcal{L}$

Halfspaces and hyperplanes

fix nonzero vector $a \in \mathbb{R}^n$ and scalar β

- **hyperplanes** and **halfspaces**, respectively, have the form

$$\mathcal{H} = \{x \mid a^T x = \beta\} \quad \text{and} \quad \mathcal{H}_- = \{x \mid a^T x \leq \beta\}$$

- a is the **normal** to the hyperplane
- hyperplanes are **affine** and **convex**
- halfspaces are convex but **not affine**

Question

Express the nonnegative orthant

$$\mathbb{R}^n_+ = \{x \mid x_i \geq 0,\ i = 1, \ldots, n\}$$

as an intersection of n halfspaces.

Convex polyhdra

$\mathcal{S}$ is a **convex polyhdron** if it's the intersection of a finite number of halfspaces:

$$\mathcal{S} = \bigcap_{i=1}^{m} \{x \mid a_i^T x \leq \beta_i\} = \{x \mid Ax \leq b\}$$

where

$$A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix} \in \mathbb{R}^{m \times n} \quad \textbf{and} \quad b \in \mathbb{R}^m$$

Question

Express the probability simplex

$$\Delta = \left\{ x \;\middle|\; \sum_{i=1}^{n} x_i = 1,\ x_i \geq 0 \right\}$$

as the intersection of n halfspaces and a hyperplane.

Convex cones

- a set $\mathcal{K} \subset \mathbb{R}^n$ is a **cone** if $x \in \mathcal{K} \iff \alpha x \in \mathcal{K}$ for all $\alpha \geq 0$
- a **convex cone** is a cone that is also convex

$$x, y \in \mathcal{K} \text{ and } \alpha, \beta \geq 0 \implies \alpha x + \beta y \in \mathcal{K}$$

Examples

$$\mathbb{R}^n_+ = \{x \mid x_i \geq 0,\ i = 1, \dots, n\} \qquad \text{(nonnegative orthant)}$$

$$\mathcal{L}^n_+ = \left\{ \begin{bmatrix} x \\ t \end{bmatrix} \in \mathbb{R}^{n+1} \;\middle|\; \|x\|_2 \leq t,\ x \in \mathbb{R}^n,\ t \in \mathbb{R} \right\} \qquad \text{(second-order cone)}$$

$$\mathcal{S}^n_+ = \{X \in \mathbb{R}^{n \times n} \mid X \succeq 0\} \qquad \text{(positive semidefinite cone)}$$

Application: Robust Portfolio Optimization

Objective: Balance risk and return by optimizing asset allocation weights w in a portfolio:

$$\max_{w} \quad \mu^T w \quad \text{subject to} \quad w^T \Sigma w \leq \sigma^2, \quad \mathbf{1}^T w = 1$$

where μ is the vector of expected returns and Σ is the covariance matrix and σ^2 is the maximum acceptable risk.

- Transform the quadratic risk constraint into a second-order cone constraint:

$$\|\Sigma^{1/2} w\|_2 \leq \sigma.$$

- **Optimization Problem:**

$$\begin{aligned} \max_{w} \quad & \mu^T w \\ \text{s.t.} \quad & \|\Sigma^{1/2} w\|_2 \leq \sigma, \\ & \mathbf{1}^T w = 1. \end{aligned}$$

- This formulation is a **second-order cone program (SOCP)** that directly maximizes return while keeping risk below a specified threshold.

Operations that preserve convexity

Let $\mathcal{C}_1, \mathcal{C}_2$ be convex sets in $\mathbb{R}^n$.

- **nonnegative scaling:**

$$\theta \mathcal{C}_1 = \{\theta x \mid x \in \mathcal{C}_1\}, \quad \theta \geq 0$$

- **intersection:**

$$\mathcal{C}_1 \cap \mathcal{C}_2$$

- **sum:**

$$\mathcal{C}_1 + \mathcal{C}_2 = \{x + y \mid x \in \mathcal{C}_1, \ y \in \mathcal{C}_2\}$$