Simplex

- assumptions
- computing feasible directions
- maintaining feasibility
- reduced costs

Assumptions

we will develop the simplex algorithm for an LP in standard form

$$\begin{array}{ll} \underset{x}{\text{minimize}} & c^T x \\ \text{subject to} & Ax = b,\ x \geq 0 \end{array}$$

where A is $m \times n$

we assume throughout this section that

- A has full row rank (no redundant rows)
- the LP is feasible
- all basic feasible solutions (ie, extreme points) are nondegenerate

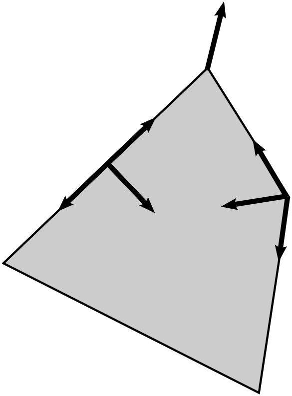
variable index sets:

- $\mathcal{B} = \{ \beta_1, \beta_2, \ldots, \beta_m \}$: basic variables
- $\mathcal{N} = \{ \eta_1, \eta_2, \ldots, \eta_{n-m} \}$: nonbasic variables

Feasible directions

a direction d is **feasible** at $x \in \mathcal{P}$ if there exists $\alpha > 0$ such that

$$x + \alpha d \in \mathcal{P}$$



Constructing feasible directions

given $x \in \mathcal{P}$ and $Ax = b$, $x \geq 0$

require for all $\alpha \geq 0$ that

$$b = A(x + \alpha d) = Ax + \alpha Ad = b + \alpha Ad$$

thus, we require $Ad = 0$

suppose that x is a basic feasible solution, so that

$$0 = Ad = \begin{bmatrix} B & N \end{bmatrix} \begin{bmatrix} d_B \\ d_N \end{bmatrix} = Bd_B + Nd_N \quad \Longrightarrow \quad Bd_B = -Nd_N$$

construct search directions by moving a **single** nonbasic variable $\eta_k \in \mathcal{N}$:

$$d_N = e_k \qquad \text{and} \qquad Bd_B = -a_{\eta_k}$$

Example

$$\begin{array}{lccccccccc} \text{minimize} & c_1x_1 & + & c_2x_2 & + & c_3x_3 & + & c_4x_4 & & \\ \text{subject to} & x_1 & + & x_2 & + & x_3 & + & x_4 & = & 2 \\ & 2x_1 & + & & + & 3x_3 & + & 4x_4 & = & 2 \\ & & & & & & & x & \geq & 0 \end{array}$$

$$\mathcal{B} = \{\, 1, 2 \,\} \quad \Longrightarrow \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \quad \Longrightarrow \quad x = \begin{bmatrix} x_B \\ x_N \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \cdots \\ 0 \\ 0 \end{bmatrix}$$

increase nonbasic variable x_3, ie, $d_N = \begin{bmatrix} d_{\eta_1} \\ d_{\eta_2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$:

$$Bd_B = -Nd_N \quad \Longrightarrow \quad \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} d_{\beta_1} \\ d_{\beta_2} \end{bmatrix} = - \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} d_{\beta_1} \\ d_{\beta_2} \end{bmatrix} = 1/2 \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

thus,

$$d = (-3/2,\ 1/2,\ 1,\ 0)$$

Change in objective

how does the objective $c^T\bar{x} = c^T(x + \alpha d)$ change as α increases?

$$\phi = c^T x \qquad \text{where } x \text{ is a basic feasible solution}$$

then

$$\begin{aligned}
\bar{\phi} &= c^T\bar{x} && (\bar{x} = x + \alpha d) \\
&= c^T(x + \alpha d) \\
&= c^T x + \alpha c^T d \\
&= \phi + \alpha \begin{bmatrix} c_B^T & c_N^T \end{bmatrix} \begin{bmatrix} d_B \\ d_N \end{bmatrix} \\
&= \phi + \alpha (c_B^T d_B + c_N^T d_N) \\
&= \phi + \alpha \; (\underbrace{c_B^T d_B + c_{\eta_p}}_{\text{reduced costs}})
\end{aligned}$$

where $d_N = e_p$, ie, only pth nonbasic variable η_p moves

Reduced costs

reduced cost for **any** variable x_j, $j = 1, \ldots, n$:

$$z_j := c_j + c_{\scriptscriptstyle B}^T d_{\scriptscriptstyle B} = c_j - c_{\scriptscriptstyle B}^T B^{-1} a_j$$

reduced costs for **basic variable** x_j, $j \in \mathcal{B}$:

$$\begin{aligned} z_j &= c_j - c_{\scriptscriptstyle B}^T B^{-1} a_j \\ &= c_j - c_{\scriptscriptstyle B}^T e_j \qquad\qquad (B^{-1}B = I \quad \Longrightarrow \quad B^{-1} a_j = e_j \text{ if } j \in \mathcal{B}) \\ &= c_j - c_j \\ &= 0 \end{aligned}$$

thus, only **nonbasic** variables need to be considered

note: if $z \geq 0$, then all feasible directions **increase** the objective

theorem: consider a BFS x with a reduced cost z.

- if $z \geq 0$ then x is optimal
- if x is optimal and nondegenerate then $z \geq 0$

Choosing a steplength

change in objective value from moving pth nonbasic variable $\eta_p \in \mathcal{N}$:

$$\bar{\phi} = \phi + \alpha z_{\eta_p}$$

$z_{\eta_p} < 0$, so choose $\alpha > 0$ as large as possible:

$$\alpha^* = \max \{ \alpha \geq 0 \mid x + \alpha d \geq 0 \}$$

case 1: if $d \geq 0$, then it is an **unbounded** feasible direction of descent, ie,

$$x + \alpha d \geq 0 \quad \text{for all} \quad \alpha \geq 0$$

case 2: if $d_j < 0$ for some j, then $x + \alpha d \geq 0$ only if

$$\alpha \leq -x_j/d_j \quad \text{for every} \quad d_j < 0$$

ratio test:

$$\alpha^* = \min_{\{j \in \mathcal{B} \mid d_j < 0\}} -\frac{x_j}{d_j}$$

Basis change

case 1: no “blocking” basic variable. Therefore d is a direction of unbounded descent

case 2: the first basic variable to “hit” a bound is “blocking”

variable swap:

- entering nonbasic variable $\eta_p \in \mathcal{N}$ becomes basic ($x_{\eta_p} \to +$)
- blocking basic variable $\beta_q \in \mathcal{B}$ becomes nonbasic ($x_{\beta_q} \to 0$)

new basic and nonbasic variables

- $\bar{\mathcal{B}} \leftarrow \mathcal{B} \setminus \{ \beta_q \} \cup \{ \eta_p \}$
- $\bar{\mathcal{N}} \leftarrow \mathcal{N} \setminus \{ \eta_p \} \cup \{ \beta_q \}$

A new basis

the new set of columns define a basic feasible solution: the new basis matrix

$$\bar{B} = [a_{\beta_1} \; a_{\beta_2} \; \cdots \; a_{\eta_p} \; \cdots \; a_{\beta_m}] \quad \text{with} \quad \eta_p \in \mathcal{N}$$

has rank m.
Note that

$$\begin{aligned} I = B^{-1}B = B^{-1} \begin{bmatrix} a_{\beta_1} & a_{\beta_2} & \cdots & a_{\beta_q} & \cdots & a_{\beta_m} \end{bmatrix} \\ = \begin{bmatrix} e_1 & e_2 & \cdots & e_q & \cdots & e_m \end{bmatrix} \end{aligned}$$

thus,

$$\begin{aligned} B^{-1}\bar{B} = B^{-1} & \begin{bmatrix} a_{\beta_1} & a_{\beta_2} & \cdots & a_{\eta_p} & \cdots & a_{\beta_m} \end{bmatrix} \\ &= \begin{bmatrix} e_1 & e_2 & \cdots & B^{-1}a_{\eta_p} & \cdots & e_m \end{bmatrix} \\ &= \begin{bmatrix} e_1 & e_2 & \cdots & -d_B & \cdots & e_m \end{bmatrix} \\ &= \begin{bmatrix} 1 & & & d_{\beta_1} & & \\ & 1 & & d_{\beta_2} & & \\ & & \ddots & \vdots & & \\ & & & d_{\beta_q} & & \\ & & & \vdots & \ddots & \\ & & & d_{\beta_m} & & 1 \end{bmatrix} \qquad \text{with } d_{\beta_q} < 0 \end{aligned}$$

Simplex without $\mathbf{B}^{-1}$

search direction: maintain $Ax = b$ and $A(x + \alpha d) = b$ for all $\alpha \geq 0$

$$Ad = 0 \quad \Longrightarrow \quad \begin{bmatrix} B & N \end{bmatrix} \begin{bmatrix} d_B \\ d_N \end{bmatrix} = 0 \quad \Longrightarrow \quad Bd_B = -Nd_N$$

effect on objective: need to choose a "good" d_N. Solve

$$B^T y = c_B \qquad \text{and} \qquad z := c - A^T y$$

for some $\alpha \geq 0$,

$$\begin{aligned} \bar{\phi} &= c^T(x + \alpha d) = c^T x + \alpha c^T d = \phi + \alpha \begin{bmatrix} c_B^T & c_N^T \end{bmatrix} \begin{bmatrix} d_B \\ d_N \end{bmatrix} \\ &= \phi + \alpha(c_B^T d_B + c_N^T d_N) \\ &= \phi + \alpha(y^T B d_B + c_N^T d_N) \\ &= \phi + \alpha(-y^T N d_N + c_N^T d_N) \\ &= \phi + \alpha(c_N - N^T y)^T d_N \\ &= \phi + \alpha z_N^T d_N \end{aligned}$$

pricing: only one nonbasic η_p moves, implying

$$d_{\scriptscriptstyle N} = e_p, \qquad Bd_{\scriptscriptstyle B} = -a_{\eta_p}, \qquad \bar{\phi} = \phi + \alpha z_{\eta_p}$$

choose p so that $z_{\eta_p} < 0$ (eg, most negative). nonbasic η_p **enters** basis

optimality: no improving direction exists if for each $j = 1, \dots, n$

$$\begin{array}{lll} x_j = 0 & \text{and} & z_j \geq 0 \\ x_j \geq 0 & \text{and} & z_j = 0 \quad \text{(must hold for basics)} \end{array}$$

ratio test: basic variable β_q **exits** basis

$$q = \underset{q \mid d_{\beta_q} < 0}{\arg\min} - \frac{x_{\beta_q}}{d_{\beta_q}},$$

new basic and nonbasic variables

- $\mathcal{B} \leftarrow \mathcal{B} \setminus \{\, \beta_q \,\} \cup \{\, \eta_p \,\}$
- $\mathcal{N} \leftarrow \mathcal{N} \setminus \{\, \eta_p \,\} \cup \{\, \beta_q \,\}$