Standard Form Polyhedra

CPSC 406 – Computational Optimization

Standard Form Polyhedra

- standard form
- converting to standard form
- two-dimensional representations
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- degeneracy: standard form

Polyhedra in standard form

Generic polyhedron

$$\mathcal{P} = \left\{ x \,\middle|\, \begin{aligned} Ax &= b \\ Cx &\leq d \end{aligned} \right\}$$

Standard-form polyhedron

$$\mathcal{P} = \left\{ x \,\middle|\, \begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned} \right\} \quad \text{with} \quad b \geq 0$$

Ensure nonnegative b

For $b_i < 0$, replace

$$a_i x = b_i \quad \longrightarrow \quad (-a_i) x = (-b_i)$$

For $d_i < 0$, replace

$$\begin{aligned} c_i^\mathsf{T} x \leq d_i \quad &\longrightarrow \quad (-c_i)^\mathsf{T} x \geq (-d_i) \\ c_i^\mathsf{T} x \geq d_i \quad &\longrightarrow \quad (-c_i)^\mathsf{T} x \leq (-d_i) \end{aligned}$$

Introduce slack and surplus variables

For every inequality constraint of the form

$$c_i^\mathsf{T} x \leq d_i \qquad (c_i^\mathsf{T} x \geq d_i)$$

introduce a new *slack* (or *surplus*) variable s_i, replacing the inequality with two constraints

$$\begin{aligned} c_i^\mathsf{T} x + s_i &= d_i \\ s_i &\geq 0 \end{aligned} \qquad \left(\begin{aligned} c_i^\mathsf{T} x - s_i &= d_i \\ s_i &\geq 0 \end{aligned}\right)$$

Reformulate free variables

- x_i is called a *free variable* if it has no constraints
- there are no free variables in standard form — every variable must be nonnegative

Converting free variables

- every free variable x_i is replaced with two new variables x'_i and x''_i, ie,

$$x_i := x'_i - x''_i, \quad x'_i \geq 0 \text{ and } x''_i \geq 0$$

- x'_i encodes the positive part of x_i
- x''_i encodes the negative part of x_i
- optimal solution necessarily has $x'_i \cdot x''_i = 0$ (why?)

Example

Consider the following LP problem:

$$\begin{aligned} \text{maximize:} \quad & 6x_1 + 8x_2 \\ \text{subject to:} \quad & 4x_1 + 2x_2 \leq 60 \\ & x_1 + 2x_2 \leq 32 \\ & x_2 \geq 5 \\ & x_1 \geq 0 \end{aligned}$$

To convert to standard form:

1. **Handle the lower bound on x_2**: Substitute $x_2 = y_2 + 5$ where $y_2 \geq 0$
2. **Introduce slack variables** for the inequality constraints
3. **Rewrite the objective** for minimization

$$\begin{aligned} \text{minimize:} \quad & -6x_1 - 8y_2 - 40 \\ \text{subject to:} \quad & 4x_1 + 2y_2 + s_1 = 50 \\ & x_1 + 2y_2 + s_2 = 22 \\ & x_1, y_2, s_1, s_2 \geq 0 \end{aligned}$$

Basic solutions in standard form

- x^* is a *basic solution* if the vectors in the basis set are linearly independent:

$$a_{i_1}, a_{i_2}, \ldots, a_{i_n}, \quad i_j \in \mathcal{B}$$

- in standard form, there are:
  - n variables ($x_1, \ldots, x_n$)
  - $m + n$ total constraints
    - m equality constraints ($Ax = b$)
    - n inequality constraints ($x \geq 0$)
- for any basic solution x:
  - the basic set $\mathcal{B}$ must have n elements
  - thus, exactly n of the constraints need to be active at x
  - m equality constraints are always satisfied
  - thus $n - m$ of the inequality constraints $x \geq 0$ should be "active"

Basic solutions in standard form

- Choosing $n - m$ of the inequality constraints to be active is the same as choosing $n - m$ variables x_i to be zero.
- Making x_i zero effectively eliminates column i from the matrix A.
- This is equivalent to choosing m columns of A! To be a basic solution, we also need these m columns to be linearly independent. So, permute the variables and partition

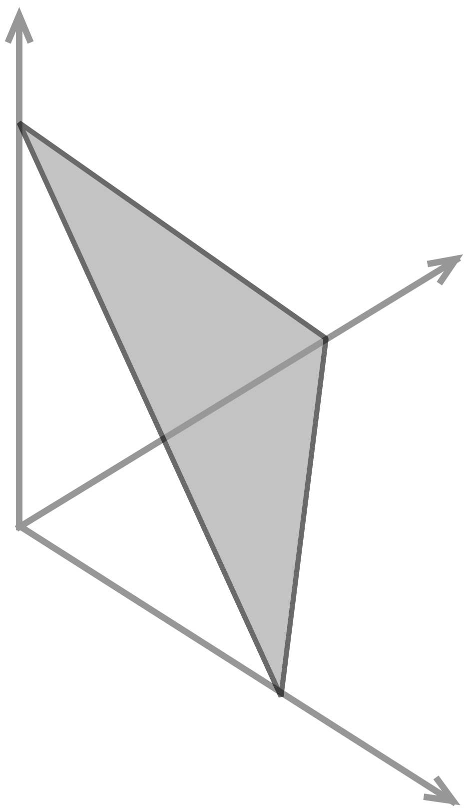
$$AP = [B \quad N] \quad \text{where } B \text{ is nonsingular}$$

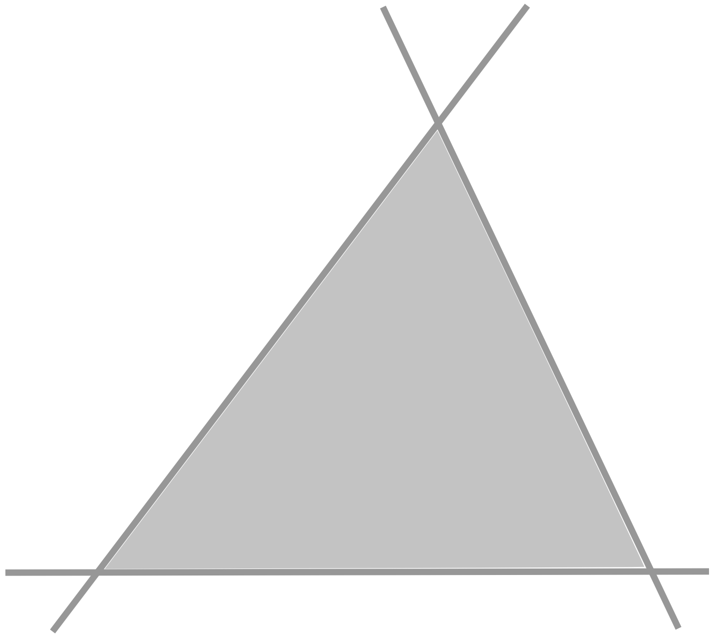
- Now we have

$$\bar{A}x = \begin{bmatrix} B & N \\ & I \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

$$\begin{aligned} x_N &= 0 \\ Bx_B &= b \end{aligned}$$

Two-dimensional representation





Degeneracy: inequality form

Polyhedron in inequality form:

$$Ax \leq b$$

A basic feasible solution x^* with

$$a_i^\intercal x^* = b_i, \quad i \in \mathcal{B} \quad \text{and} \quad a_i^\intercal x^* < b_i, \quad i \notin \mathcal{B}$$

is *degenerate* if # of indices in $\mathcal{B}$ is greater than n

- property of the *description* of the polyhedron
- affects the performance of some algorithms
- disappears for small perturbations of b

Degeneracy: standard form

Polyhedron in standard form:

$$Ax = b, \quad x \geq 0$$

A basic solution partitions the variables into two sets:

$$\begin{bmatrix} B & N \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} = b \quad \text{with} \quad x_N = 0$$

ie,

$$Bx_B = b$$

A basic feasible solution in standard form is *degenerate* if more than $n - m$ components in x are zero, ie,

$$x = \begin{bmatrix} x_B \\ x_N \end{bmatrix} \begin{matrix} m \\ n-m \end{matrix} = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix} \longleftarrow \text{\small has some zero components}$$