SVD

CPSC 406 – Computational Optimization

Overview

The singular value decomposition (SVD) reveals many of the most important properties of a matrix.
It generalizes the eigenvalue decomposition to non-square matrices.

- geometric interpretation
- reduced SVD
- full SVD
- formal definition

Matrix Rank

The **rank** of a matrix A is the maximum number of linearly independent columns (or rows) of A. It indicates the dimension of the subspace spanned by its columns (or rows).

- **Column rank** is the dimension of the column space $\mathbf{range}(A)$
- **Row rank** is the dimension of the row space $\mathbf{range}(A^T)$
- **Full rank** if $\mathbf{rank}(A) = \min(m, n)$
- for any matrix, the row and column rank are equal, so we just say **rank**

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

Implications

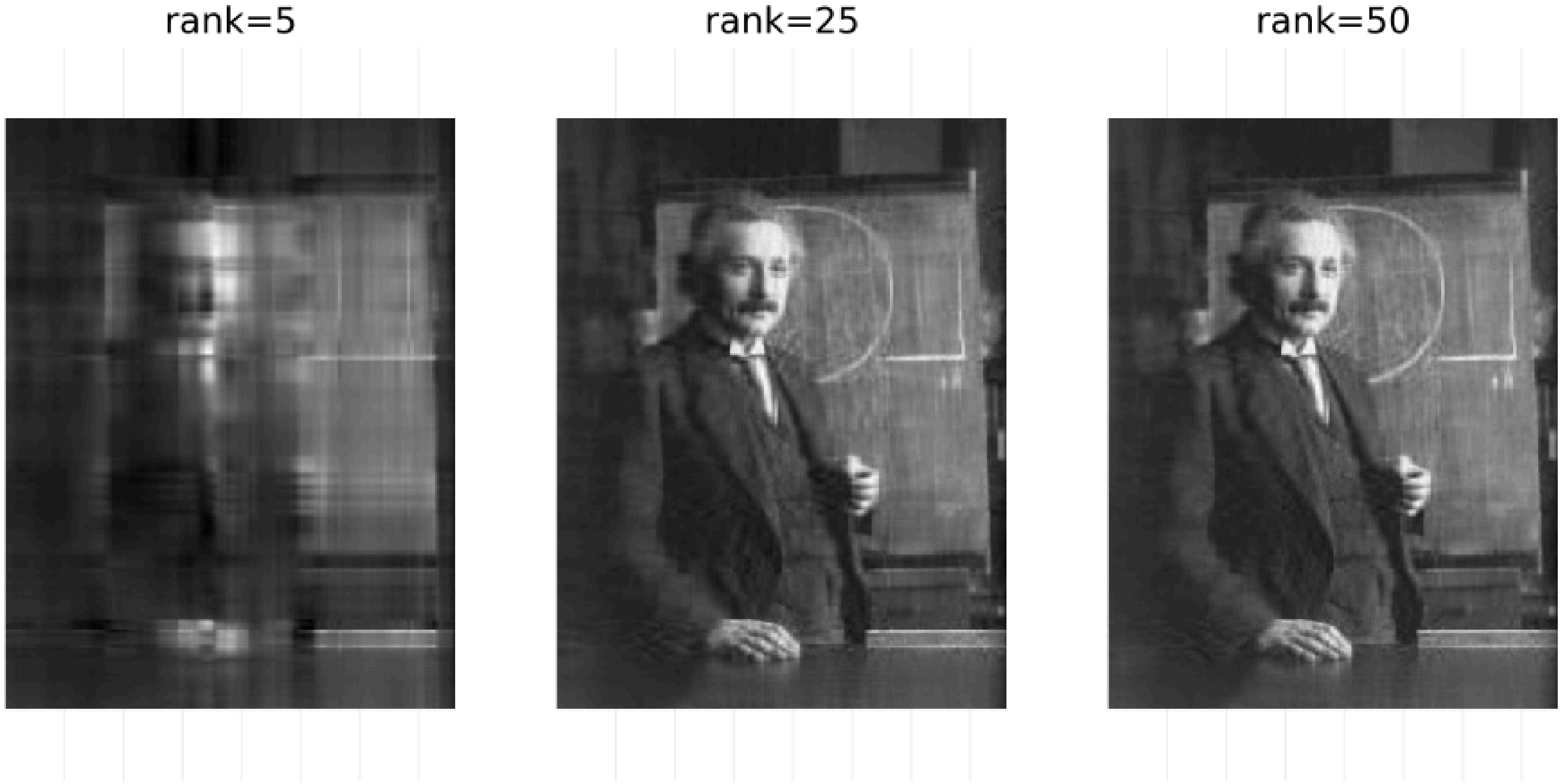
- higher rank means more linearly independent vectors $\Rightarrow$ more information or dimensions represented by the matrix
- for a square matrix, rank is the number of non-zero eigenvalues

Multiplication Table is Rank 1

```
u = collect(1:12)
A = u*u'
```

```
12×12 Matrix{Int64}:
  1   2   3   4   5   6   7   8    9   10   11   12
  2   4   6   8  10  12  14  16   18   20   22   24
  3   6   9  12  15  18  21  24   27   30   33   36
  4   8  12  16  20  24  28  32   36   40   44   48
  5  10  15  20  25  30  35  40   45   50   55   60
  6  12  18  24  30  36  42  48   54   60   66   72
  7  14  21  28  35  42  49  56   63   70   77   84
  8  16  24  32  40  48  56  64   72   80   88   96
  9  18  27  36  45  54  63  72   81   90   99  108
 10  20  30  40  50  60  70  80   90  100  110  120
 11  22  33  44  55  66  77  88   99  110  121  132
 12  24  36  48  60  72  84  96  108  120  132  144
```

Low-rank image approximation



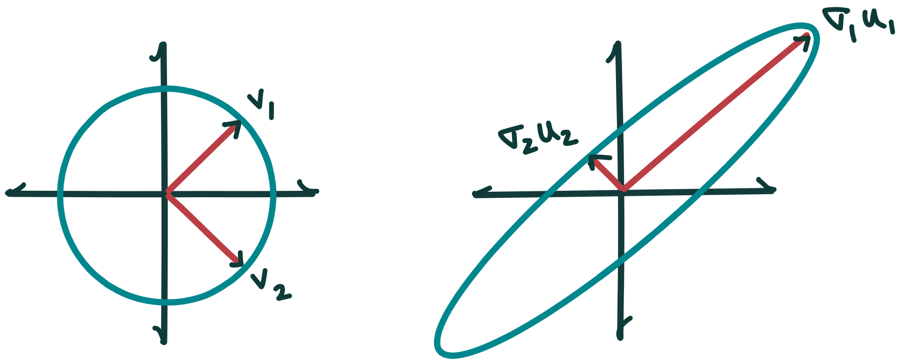
Singular value decomposition

For **any** $m \times n$ matrix A with rank r

$$A = U\Sigma V^T = [u_1 \mid u_2 \mid \cdots \mid u_r] \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix} \begin{bmatrix} v_1^T \\ \hline v_2^T \\ \hline \vdots \\ \hline v_r^T \end{bmatrix} = \sum_{j=1}^{r} \sigma_j u_j v_j^T$$

left U and **right** V singular vectors are orthonormal and **singular values**:

$$U^T U = I_r, \qquad V^T V = I_r, \qquad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$$

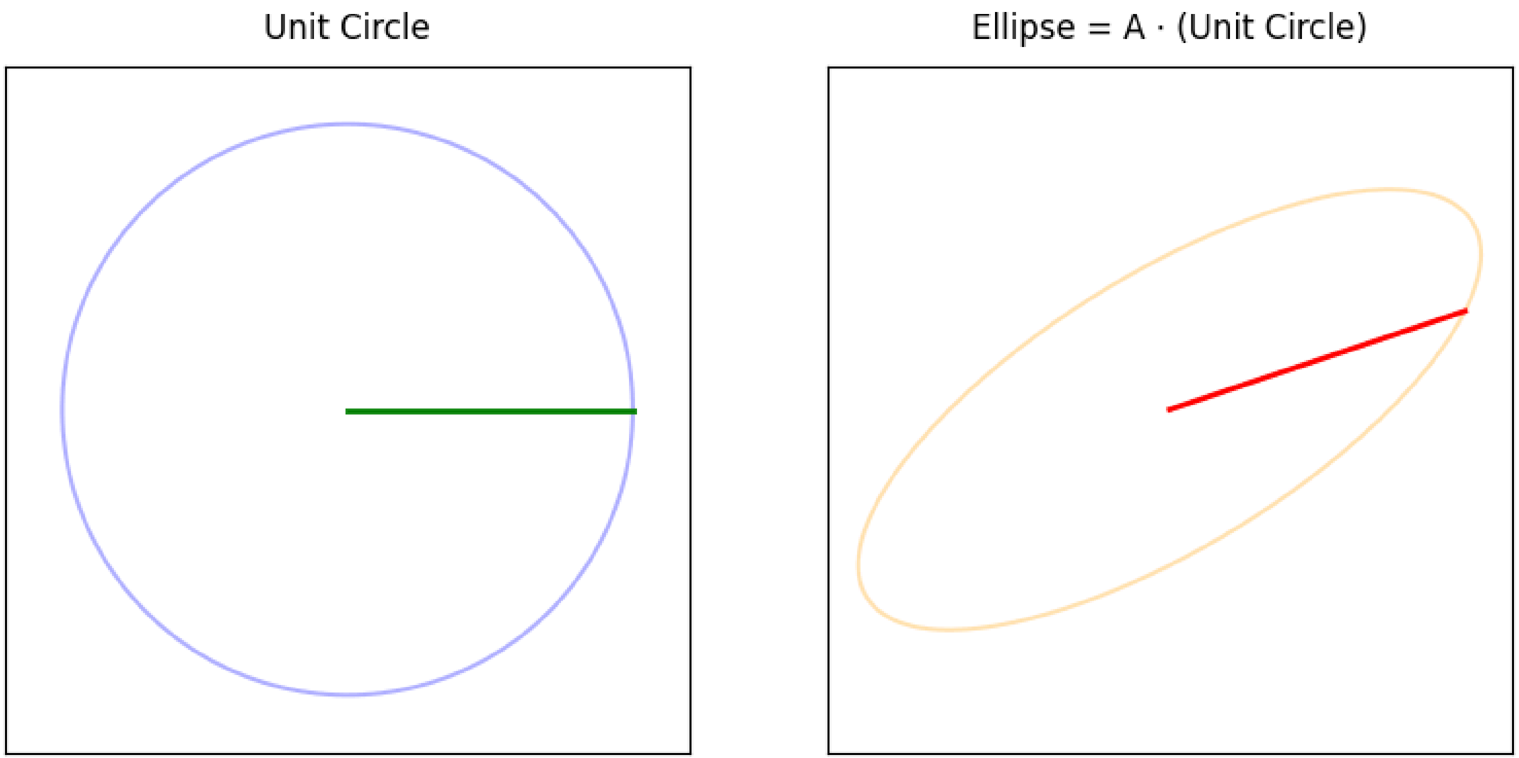


for $j = 1, \ldots, r$

$$AV = U\Sigma \qquad A^T U = V\Sigma$$

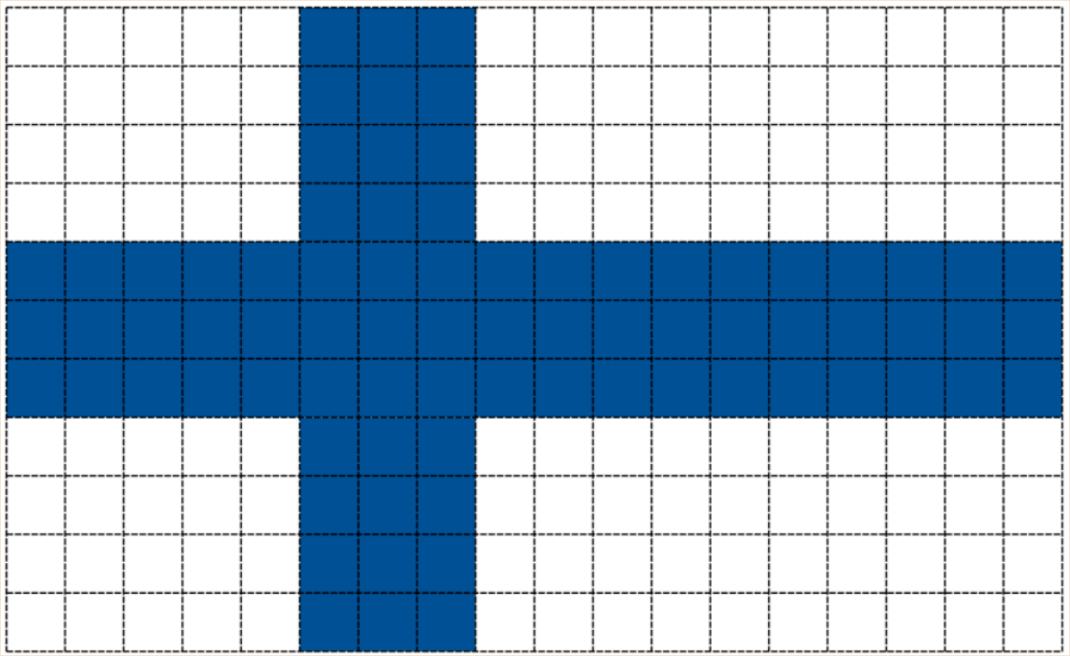
$$Av_j = \sigma_j u_j \qquad A^T u_j = \sigma_j v_j$$

SVD Visualization



Question: Flag rank

What is the rank of the Finnish flag?



a. 1
b. 2
c. 3
d. 4
e. 5

Question: Flag rank II

What is the rank of the Greek flag?

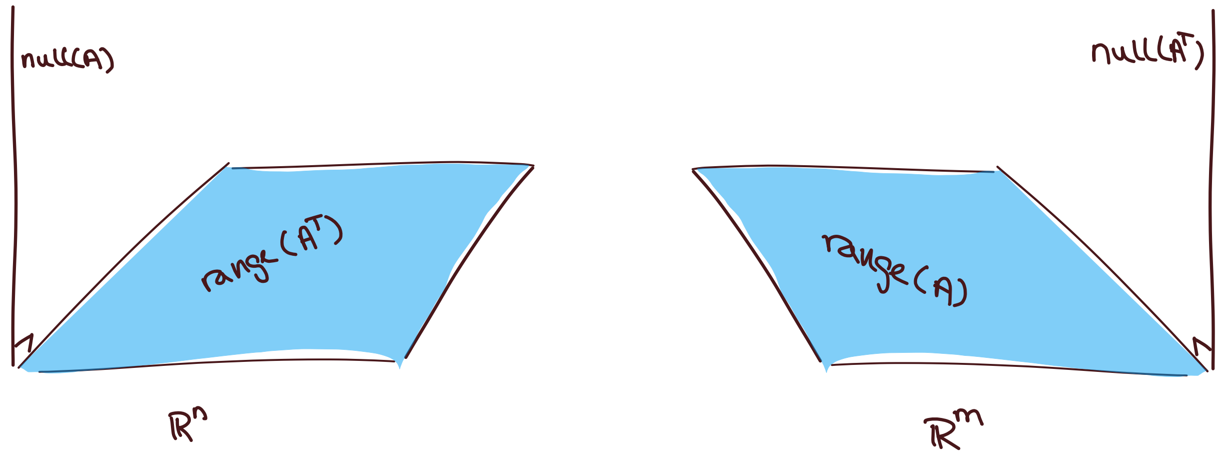
a. 1
b. 2
c. 3
d. 4
e. 5

Column and Row Bases

The left- and right- singular vectors constitute orthogonal bases for the four fundamental subspaces of A

$$\mathbf{proj}_{\mathbf{range}(A^T)} = VV^T \qquad \mathbf{proj}_{\mathbf{range}(A)} = UU^T$$

$$\mathbf{proj}_{\mathbf{null}(A)} = I_n - VV^T \qquad \mathbf{proj}_{\mathbf{null}(A^T)} = I_m - UU^T$$



SVD Construction

Conceptually, we can construct the SVD from the Grammian $A^T A$:

1. gather the **eigenvalues** of $A^T A$ in descending order (may be multiplicity):

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0, \qquad \Lambda := \mathbf{Diag}(\lambda_1, \ldots, \lambda_r)$$

2. because $A^T A$ is symmetric, the **spectral theorem** ensures it's diagonalizable

$$A^T A = V \Lambda V^T$$

3. define **singular values**:

$$\sigma_i := \sqrt{\lambda_i}, \qquad \Sigma := \mathbf{Diag}(\sigma_1, \ldots, \sigma_r)$$

4. define left **singular vectors** (deduce that U is orthonormal):

$$U := A V \Sigma^{-1}$$

5. summary:

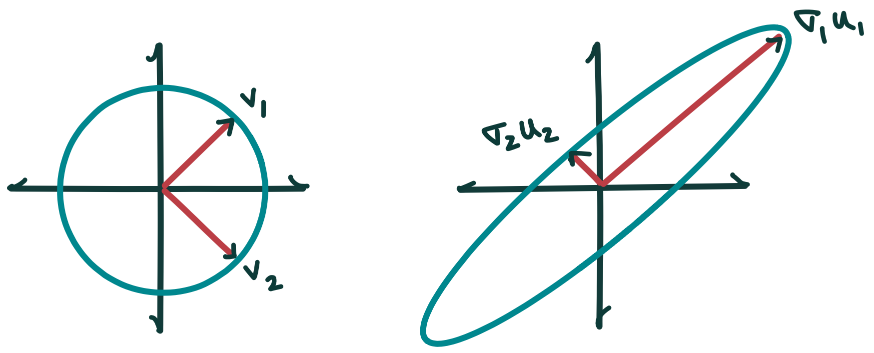
$$A = U \Sigma V^T$$

Properties

Given SVD $A = U\Sigma V^T$ with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$

1. U is a basis for $\mathbf{range}(A)$
2. V is a basis for $\mathbf{range}(A^T)$
3. spectral norm of A:

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2 = \sigma_1$$



4. Frobenious norm of A:

$$\|A\|_F := \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2} \equiv \sqrt{\mathbf{trace}(A^T A)} = \sqrt{\sum_{i=1}^{r} \sigma_i^2}$$

Question: Spectral norm I

Suppose that $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$ are unit-norm vectors. Then the outer product uv^T is an $m \times n$ matrix with rank 1. What is the spectral norm of uv^T?

a. 1

b. $u^T v$

c. 2

d. $||u||_2 + ||v||_2$

Question: Spectral norm II

Suppose that $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ are nonzero vectors. Then the outer product xy^T is an $m \times n$ matrix with rank 1. What is the spectral norm of xy^T?

a. 1

b. $x^T y$

c. $||x||_2 \cdot ||y||_2$

d. $||x||_2 + ||y||_2$

Low-rank approximation

The SVD decomposes any matrix A with rank r into a sum of rank-1 matrices:

$$A = U\Sigma V^T = [u_1 \mid u_2 \mid \cdots \mid u_r] \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix} \begin{bmatrix} v_1^T \\ \hline v_2^T \\ \hline \vdots \\ \hline v_r^T \end{bmatrix} = \sum_{j=1}^{r} \sigma_j u_j v_j^T$$

Eckart-Young Theorem

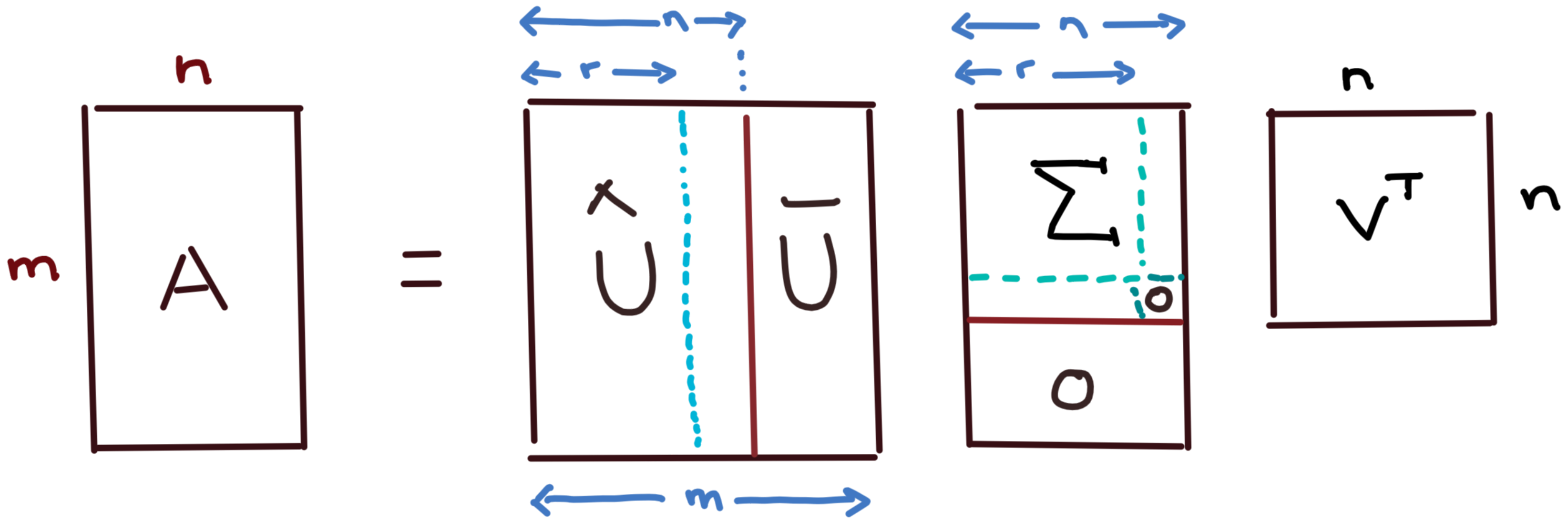
The best rank-k approximation to A is given by rank-k approximation

$$A_k = \sum_{j=1}^{k} \sigma_j u_j v_j^T$$

with error $E_k := A - A_k$ satisfying

$$\|E_k\|_2 = \sigma_{k+1} \quad \text{and} \quad \|E_k\|_F = \sqrt{\sum_{j=k+1}^{r} \sigma_j^2}$$

Full SVD



$$U = [\,\hat{U} \mid \bar{U}\,]$$
$$\hat{U} = [\,u_1, \ldots, u_r, u_{r+1}, \ldots, u_n\,]$$
$$\bar{U} = [\,u_{n+1}, \ldots, u_m\,]$$
$$V = [\,v_1, \ldots, v_n\,]$$
$$\Sigma = \mathbf{Diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0)$$

Provides **orthogonal bases** for all four fundamental subspaces

$$\mathbf{range}(A) = \mathbf{span}\{u_1, \ldots, u_r\}$$
$$\mathbf{null}(A^T) = \mathbf{span}\{u_{r+1}, \ldots, u_m\}$$
$$\mathbf{range}(A^T) = \mathbf{span}\{v_1, \ldots, v_r\}$$
$$\mathbf{null}(A) = \mathbf{span}\{v_{r+1}, \ldots, v_n\}$$

Minimum norm least-squares solution

If A is $m \times n$ with $\mathbf{rank}(A) = r < n$, then infinitely many least-squares solutions:

$$\mathcal{X} = \{x \in \mathbb{R}^n \mid A^T A x = A^T b\}$$

SVD provides the **minimum norm** solution $\bar{x} = \min\{\|x\| \mid x \in \mathcal{X}\}$

If $A = U\Sigma V^T$ is the full SVD of A, then

$$\begin{aligned}
\|Ax - b\|^2 &= \|(U^T A V)(V^T x) - U^T b\|^2 \\
&= \|\Sigma y - U^T b\|^2 && (y := V^T x) \\
&= \sum_{j=1}^{r} (\sigma_j y_j - \bar{b}_j)^2 + \sum_{j=r+1}^{n} \bar{b}_j^2 && (\bar{b}_j = u_j^T b)
\end{aligned}$$

Choose

$$y_j = \begin{cases} \bar{b}_j/\sigma_j & j = 1:r \\ 0 & j = r+1:n \end{cases} \quad \Longrightarrow \quad \bar{x} = Vy = \sum_{j=1}^{r} \frac{u_j^T b}{\sigma_j} v_j$$

Pseudoinverse of a Matrix

The **pseudoinverse** of a matrix A, denoted A^+, generalizes the inverse for matrices that may not be square or full rank. If $A = U\Sigma V^T$ is the SVD of A, then:

$$A^+ := V\Sigma^+U^T \quad \text{where} \quad \Sigma_{ij}^+ = \begin{cases} 1/\sigma_i & \text{if } \sigma_i > 0 \\ 0 & \text{if } \sigma_i = 0 \end{cases}$$

Key Properties

1. A^+A projects onto the **column space** of A
2. AA^+ projects onto the **row space** of A^T
3. A^+ always exists for any matrix A, regardless of its shape or rank

Connection to the Minimum Norm Solution

The pseudoinverse provides the **minimum norm least-squares solution** to $\|Ax - b\|$:

$$\bar{x} = A^+b$$